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Convergence in distribution of fuzzy random variables in L^p-type metrics

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Abstract

General properties of convergence in distribution for fuzzy random variables are studied as regards its interplay with the structure of the space of fuzzy sets. In particular, its behaviour with respect to taking tuples of fuzzy random variables, adding, multiplying by a scalar, taking the union, preserving inclusion ordering, and subsuming convergence in distribution of random sets is established.

Keywords: Convergence in distribution; Fuzzy random variable; Slutski theorem; Weak convergence

1. Introduction

By a result of Krätschmer (see Proposition 2.2 below), fuzzy random variables can be seen as random elements of a space formed by generalized fuzzy numbers and endowed with the separable L^p-type metric d_p, for any $p \in [1, \infty)$. That means that fuzzy random variables are amenable to the methods of probability in metric spaces [7,19] and in particular convergence in distribution can be defined via weak convergence of the induced probability measures. This definition does not rely on cumulative distribution functions (while being equivalent to the ordinary definition in $\mathbb{R}^d$) and so can be applied in more general spaces.

In recent papers [1,2] we have studied some properties of fuzzy random variables under that kind of convergence, like a Skorokhod theorem and several applications (continuous mapping theorem, Vitali convergence theorem, dominated convergence theorem, existence of extensions, perfect distributions). We have also shown that it behaves adequately with respect to some aspects of the structure of spaces of fuzzy sets [3], as will be described in Section 3.

In this paper, we undertake a more systematic study of its properties. First we will study 'vectors' or tuples of fuzzy random variables, which will be needed for the subsequent work. Indeed, if we want to study the joint convergence in distribution of two (or several) fuzzy random variables, the joint distribution is defined on a Cartesian product of spaces of fuzzy sets to which known results for fuzzy random variables cannot be immediately applied. We need

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to establish that the joint object (X, Y) can meaningfully be identified with its (fuzzy) Cartesian product $X \times Y$, that this defines a fuzzy random variable in a space of fuzzy sets of higher dimension, and that joint convergence $(X_n, Y_n) \to (X, Y)$ is consistent with convergence $X_n \times Y_n \to X \times Y$ in that larger space.

Then we will study the relationships between convergence in distribution and the operations of sum, product by a scalar, and union. The intersection of two fuzzy sets can be non-normal while the d_p-metrics are only defined for normal fuzzy sets, whence we only consider the union. To mention a representative example, we will show that $(X_n, Y_n) \to (X, Y)$ in distribution implies $X_n + Y_n \to X + Y$ in distribution, with analogous results for the other two operations. From them, we will obtain a version of the Slutski theorem for fuzzy random variables. This is a basic convergence result which, loosely speaking, states that a slight perturbation of a convergent sequence still converges (it is used to prove, for instance, that Student's t-statistic is asymptotically normal).

We will also study the relationships with the inclusion between fuzzy sets. For random variables, convergence in distribution is too weak to have nice order-preserving properties: if a sequence converges to a normal distribution X with zero mean, since $-X$ is identically distributed to X it will also be a limit in distribution of the sequence but the transformation $x \mapsto -x$ reverses the order. The same happens with the inclusion order here. We will show though that $X_n \subseteq U$ for all n, and $X_n \to X$ in distribution, imply $X \subseteq U$ for a fixed fuzzy set U, and that convergence in probability is strong enough to let us replace U by a fuzzy random variable Y.

It will be established that convergence in distribution of random (compact convex) sets is consistent with that of their indicator functions when regarded as fuzzy random variables. Finally, we show that convergence in distribution can be characterized using d_p-continuous transformations from fuzzy sets to fuzzy sets.

The structure of the paper is as follows. Sections 2 and 3 contain the preliminaries and recap prior results on convergence in distribution. In Section 4, the connection with Cartesian products is studied. Section 5 is devoted to the relationships with the operations between fuzzy sets, while the results about inclusion, random sets and d_p-continuous mappings are in Section 6. The paper concludes with some final remarks in Section 8.

2. Preliminaries

Let $\mathbb{E}$ be a topological space. We denote by $\mathcal{B}_{\mathbb{E}}$ its *Borel σ-algebra*, i.e., the σ-algebra generated by its open sets. A Borel measurable mapping with values in $\mathbb{E}$ will be generally called a *random element* of $\mathbb{E}$. We will denote $(\Omega, \mathcal{A}, P)$ the general probability space where the random elements are defined. The Lebesgue measure in [0, 1] will be denoted by ℓ. The indicator function of a set $K \subseteq \mathbb{R}^d$ will be denoted by I_K, and its closure by $\operatorname{cl} K$. Denote by B the closed unit ball in $\mathbb{R}^d$.

Let X_n, X be random elements in a topological space $\mathbb{E}$. Then $\{X_n\}_n$ *converges weakly* to X if $E[f(X_n)] \to E[f(X)]$ for every continuous bounded function $f : \mathbb{E} \to \mathbb{R}$ (see [7, Chapter 1] for more details).

A topological space $\mathbb{E}$ is *Polish* if its topology is generated by some complete separable metric, *Lusin* if it is the continuous image of a Polish space by a bijective mapping and *Suslin* if it is the continuous image of a Polish space. A probability measure is *Radon* if $P(A) = \sup_{K \subseteq A} P(K)$ where K ranges over compact sets. Hence a topological space $\mathbb{E}$ is *Radon* if every probability measure P in $\mathbb{E}$ is a Radon measure.

Let $\mathcal{K}(\mathbb{R}^d)$ be the space of all non-empty compact subsets of $\mathbb{R}^d$ and consider its subspace $\mathcal{K}_c(\mathbb{R}^d)$, which contains all non-empty compact convex subsets of $\mathbb{R}^d$.

The *Hausdorff metric* in $\mathcal{K}(\mathbb{R}^d)$ is defined by

$$d_H(K, L) = \max\{\sup_{x \in K} \inf_{y \in L} ||x - y||, \sup_{y \in L'} \inf_{x \in K} ||x - y||\}$$
$$= \inf\{\varepsilon > 0 : K \subseteq L + \varepsilon B, L \subseteq K + \varepsilon B\}.$$

The *norm* or *magnitude* of K is

$$\|K\| = d_H(K, \{0\}).$$

The Hausdorff metric has the following property in the space $\mathcal{K}_c(\mathbb{R}^d)$.

Lemma 2.1. *Let $K \in \mathcal{K}_c(\mathbb{R}^d)$ and $a, b \in \mathbb{R}$. Then*

$$d_H(aK, bK) \le |a - b| \cdot \|K\|.$$

Proof. It is immediate if $a = 0$ or $b = 0$. Without loss of generality, assume $a \geq b$. If $b > 0$,

$$d_H(aK, bK) = d_H((a-b)K + bK, bK) \leq d_H((a-b)K, \{0\}) = |a-b|\|K\|,$$

since d_H is translation invariant in $\mathcal{K}_c(\mathbb{R}^d)$. If both a and b are negative, we can apply the same reasoning. Next, if $a > 0$ and $b < 0$ we have

$$d_H(aK, bK) \leq d_H(aK, \{0\}) + d_H(\{0\}, bK)$$
$$= a\|K\| + (-b)\|K\| = (a-b)\|K\| \leq |a-b|\|K\|. \quad \square$$

A mapping $X : \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ is a *random set* (also a *random compact convex set* in the literature) if X is measurable with respect to the Borel σ-algebra $\mathcal{B}_{\mathcal{K}_c(\mathbb{R}^d)}$ generated by the topology of the Hausdorff metric.

Denote by τ_F the *Fell topology* in the space of non-empty closed subsets of $\mathbb{R}^d$. It is defined by its subbase formed by all sets of closed sets having the form $\{A : A \cap K = \emptyset\}$ or $\{A : A \cap G \neq \emptyset\}$, where K and G are respectively compact and open.

Let $\mathcal{F}(\mathbb{R}^d)$ be the space of functions (fuzzy sets) $U : \mathbb{R}^d \to [0, 1]$ whose α-cuts ($\alpha \in [0, 1]$) are in $\mathcal{K}(\mathbb{R}^d)$. Let $\mathcal{F}_c(\mathbb{R}^d)$ be the space of fuzzy subsets of $\mathbb{R}^d$, i.e., functions $U : \mathbb{R}^d \to [0, 1]$ whose α-cuts ($\alpha \in [0, 1]$) are in $\mathcal{K}_c(\mathbb{R}^d)$.

Recall that the α-cuts of a fuzzy set U are

$$U_\alpha = \{x \in \mathbb{R}^d : U(x) \geq \alpha\}$$

for each $\alpha \in (0, 1]$, and U_0 denotes the closure of its support.

Consider the space

$$\widehat{\mathcal{F}}_{c,p}(\mathbb{R}^d) = \{U : \mathbb{R}^d \to [0,1] : U_\alpha \in \mathcal{K}_c(\mathbb{R}^d)\, \forall \alpha \in (0,1], \left[\int_{(0,1]} (d_H(U_\alpha, \{0\}))^p \, d\alpha\right]^{1/p} < \infty\}.$$

For each $p \in [1, \infty)$, the metric d_p in $\widehat{\mathcal{F}}_{c,p}(\mathbb{R}^d)$ and $\mathcal{F}(\mathbb{R}^d)$, is defined by

$$d_p(U, V) = \left[\int_{(0,1]} (d_H(U_\alpha, V_\alpha))^p \, d\alpha\right]^{1/p}.$$

By [16, Corollary 3.3], $(\widehat{\mathcal{F}}_{c,p}(\mathbb{R}^d), d_p)$ is complete and separable and is a completion of $(\mathcal{F}_c(\mathbb{R}^d), d_p)$.

The *norm* of a fuzzy set $U \in (\widehat{\mathcal{F}}_{c,p}(\mathbb{R}^d), d_p)$ is

$$\|U\|_p = d_p(U, I_{\{0\}}).$$

A mapping $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ is called a *fuzzy random variable* if, for each $\alpha \in [0, 1]$, the α-cut mapping $X_\alpha : \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ defined by $X_\alpha(\omega) = (X(\omega))_\alpha$ for each $\omega \in \Omega$ is a random set (see [20]). Within this paper, we will identify fuzzy random variables with random elements of $\mathcal{F}_c(\mathbb{R}^d)$, using the following result [15, Theorem 6.6.(i), part (iii)⇔(iv)].[1]

Proposition 2.2. (Krätschmer) *Let $p \in [1, \infty)$. A mapping $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ is a fuzzy random variable if and only if it is a random element of the space $(\mathcal{F}_c(\mathbb{R}^d), d_p)$.*

We will denote by σ_L the natural σ-algebra in $\mathcal{F}_c(\mathbb{R}^d)$ with which a mapping is a fuzzy random variable if and only if it is measurable, i.e., the smallest σ-algebra that makes the mappings $U \in \mathcal{F}_c(\mathbb{R}^d) \mapsto U_\alpha \in \mathcal{K}_c(\mathbb{R}^d)$ measurable. Thus σ_L is the Borel σ-algebra induced by the metric d_p in $\mathcal{F}_c(\mathbb{R}^d)$, for each $p \in [1, \infty)$.

[1] Notice that part (iv) of that result considers mappings with values in the larger space $\mathcal{F}(\mathbb{R}^d)$ instead of $\mathcal{F}_c(\mathbb{R}^d)$. Thus our definition demands measurability of each level mapping X_α as a random set with compact convex values while part (iv) considers just compact values. Both are well known to be equivalent since the space of compact convex sets is closed (in the Hausdorff metric) in the space of compact sets.

A fuzzy random variable X is called *integrably bounded* if $E[\|X_0\|] < \infty$. Then the *expectation* of X is the unique fuzzy set $E[X] \in \mathcal{F}_c(\mathbb{R}^d)$ such that

$$(E[X])_\alpha = E_A[X_\alpha]$$

for each $\alpha \in [0, 1]$ (see [20]), where

$$E_A[X_\alpha] = \{E[f] : f : \Omega \to \mathbb{R},\ f \in L^1(\Omega, \mathcal{A}, P),\ f \in X_\alpha\ P\text{-a.s.}\}$$

is the *Aumann expectation* of X_α, which is a compact convex set.

A sequence of probability measures $\{P_n\}_n$ on σ_L is said to *converge weakly in* d_p to a probability measure P if

$$\int f\,dP_n \to \int f\,dP$$

for every $f : \mathcal{F}_c(\mathbb{R}^d) \to \mathbb{R}$ which is d_p-continuous and bounded. More generally, let $(\mathbb{E}, d)$ be a metric space and let X_n, X be random elements in $\mathbb{E}$. Then X_n *converges weakly* to X if $E[f(X_n)] \to E[f(X)]$ for every continuous bounded function $f : \mathbb{E} \to \mathbb{R}$.

Lemma 2.3. *[7, Theorem 2.1] Let $\mathbb{E}$ be a metric space, P a probability measure and $\{P_n\}_n$ a sequence of probabilities in $(\mathbb{E}, \mathcal{B}_{\mathbb{E}})$. Then the following conditions are equivalent:*

- *$P_n \to P$ weakly,*
- *For every open set G we have $\liminf_{n\to\infty} P_n(G) \geq P(G)$,*
- *For every closed set F we have $\limsup_{n\to\infty} P_n(F) \leq P(F)$.*

A sequence $\{X_n\}_n$ of fuzzy random variables *converges in distribution in* d_p to a fuzzy random variable X if their distributions P_{X_n} converge weakly to P_X, or equivalently if $X_n \to X$ weakly as random elements of $(\mathcal{F}_c(\mathbb{R}^d), d_p)$. It *converges almost surely in* d_p to X if $P(d_p(X_n, X) \to 0) = 1$. It *converges in probability in* d_p if $P(d_p(X_n, X) < \varepsilon) \to 1$ for each $\varepsilon > 0$.

A function f between metric spaces $\mathbb{E}$ and $\mathbb{F}$ is *bounded* if and only if $f(\mathbb{E})$ is contained in a ball of $\mathbb{F}$.

For any $U \in \mathcal{F}_c(\mathbb{R}^d)$, denote by $\operatorname{end} U$ its *endograph*, i.e.,

$$\operatorname{end} U = \{(x, \alpha) \in \mathbb{R}^d \times [0, 1] : U(x) \geq \alpha\}.$$

In [11], the following results are proven in a more general setting including the space $\mathcal{F}_c(\mathbb{R}^d)$, where we will enunciate them for simplicity.

Theorem 6.4, (i) $\Leftrightarrow$ (ii) in [11] establishes a relationship between convergence of α-cuts of a sequence of fuzzy sets and convergence of their endographs.

Lemma 2.4. *Let $U_n, U \in \mathcal{F}_c(\mathbb{R}^d)$ such that $(U_n)_\alpha, U_\alpha$ are bounded sets for each $\alpha \in (0, 1]$. Then $d_H(\operatorname{end} U, \operatorname{end} U_n) \to 0$ if and only if $d_H(U_\alpha, (U_n)_\alpha) \to 0$ for almost every $\alpha \in (0, 1)$.*

Next, [11, Theorem 6.6] states that convergence in d_p is stronger than convergence of endographs of fuzzy sets.

Lemma 2.5. *Let $U_n, U \in \mathcal{F}_c(\mathbb{R}^d)$. If $d_p(U_n, U) \to 0$, then $d_H(\operatorname{end} U_n, \operatorname{end} U) \to 0$.*

As a consequence, we establish the following result.

Corollary 2.6. *Let $U_n, U \in \mathcal{F}_c(\mathbb{R}^d)$. If $d_p(U_n, U) \to 0$, then $d_H((U_n)_\alpha, U_\alpha) \to 0$ for almost every $\alpha \in (0, 1)$.*

Finally, we will state some results related to the product of fuzzy sets and the operations that will be used in Section 4. Let $U \in \mathcal{F}_c(\mathbb{R}^d)$ and $V \in \mathcal{F}_c(\mathbb{R}^{d'})$ and denote by $U \times V \in \mathcal{F}_c(\mathbb{R}^{d+d'})$ the *Cartesian product* given by

$$U \times V : (x, y) \in \mathbb{R}^{d+d'} \mapsto (U \times V)(x, y) = \min\{U(x), V(y)\}.$$

Lemma 2.7. *Let $U, V \in \mathcal{F}_c(\mathbb{R}^d)$. Then*

$$(U \times V)_\alpha = U_\alpha \times V_\alpha.$$

Lemma 2.8. *[5, Theorem 1.12.15, p. 66] Let $K, K', L, L' \in \mathcal{K}_c(\mathbb{R}^d)$. Then*

$$d_H(K \cup K', L \cup L') \leq \max\{d_H(K, L), d_H(K', L')\}.$$

Let $U, V \in \mathcal{F}_c(\mathbb{R}^d)$. Then the *union* of U and V is defined as

$$(U \cup V)(x) = \max\{U(x), V(x)\}$$

for every $x \in \mathbb{R}^d$.

Lemma 2.9. *Let $U, V \in \mathcal{F}_c(\mathbb{R}^d)$. Then for every $\alpha \in [0, 1]$,*

$$(U \cup V)_\alpha = U_\alpha \cup V_\alpha.$$

3. Previous results

For the reader's benefit, this section collects the main results obtained so far about convergence in distribution in the d_p-metrics, some of which will be used in the sequel. These results suggest that the definition of convergence in distribution via the general theory of weak convergence of probability measures in metric spaces is viable and suitable for fuzzy random variables, and justify the systematic study of its properties that we undertake here.

For completeness, we mention that some results exist for d_∞ and the Skorokhod metric. Joo *et al.* [13] studied several characterizations of tightness under the Skorokhod metric, which is a standard way of proving convergence in distribution. Terán [27, Proposition 10] presented a Skorokhod theorem in the metric d_∞ as an application of the Skorokhod theorem in metric spaces, while Alonso de la Fuente and Terán [2] proved Vitali convergence theorems and dominated convergence theorems under the assumption of weak convergence.

An analogous proof to that in [1, Theorem 5.1] yields the following form of the continuous mapping theorem, which will be used later on.

Lemma 3.1. *Let $(\mathbb{E}, d)$ be a metric space. Let X_n and X be fuzzy random variables such that $X_n \to X$ in distribution in d_p. If $f : (\mathcal{F}_c(\mathbb{R}^d), d_p) \to \mathbb{E}$ is a P_X-almost surely continuous function, then $f(X_n) \to f(X)$ in distribution in d.*

In [1, Theorem 3.5], we proved a version of the Skorokhod representation theorem for fuzzy random variables in $(\mathcal{F}_c(\mathbb{R}^d), d_p)$.

Lemma 3.2. *Let $p \in [1, \infty)$. Let P_n, P be probability measures on σ_L, such that $P_n \to P$ in distribution. Then there exist fuzzy random variables $X_n, X : ([0, 1], \mathcal{B}_{[0,1]}, \ell) \to (\mathcal{F}_c(\mathbb{R}^d), d_p)$, such that*

(a) *The distributions of X_n and X are P_n and P, respectively.*
(b) *$X_n(t) \to X(t)$ in d_p for every $t \in [0, 1]$.*

The following dominated convergence theorem under the assumption of convergence in distribution appears in [1, Theorem 4.7]. More related results can be found in [2].

Lemma 3.3. *Let X_n and X be integrably bounded fuzzy random variables. If $X_n \to X$ in distribution in d_p and there exists $g \in L^1(\Omega, \mathcal{A}, P)$ such that $d_p(X_n, I_{\{0\}}) \leq g$ for all $n \in \mathbb{N}$, then $E[X_n] \to E[X]$ in d_p.*

Convergence in distribution can be studied by embedding the fuzzy sets into an L^p-type function space [3, Theorem 1].

Theorem 3.4. *Let $p \in [1, \infty)$. Let X_n, X be fuzzy random variables. Then the following conditions are equivalent.*

1. $X_n \to X$ *in distribution in* $(\mathcal{F}_c(\mathbb{R}^d), d_p)$.
2. $s_{X_n} \to s_X$ *in distribution in* $L^p(\mathbb{S}^{d-1} \times [0,1], \mathcal{B}_{\mathbb{S}^{d-1}} \otimes \mathcal{B}_{[0,1]}, \lambda \otimes \ell)$,

where λ *denotes the uniform probability distribution in* $\mathbb{S}^{d-1}$.

In the case of random trapezoidal fuzzy sets $Tra(\xi_1, \xi_2, \xi_3, \xi_4)$, defined as

$$X(\omega)(x) = \begin{cases} 0 & \text{if } x < \xi_1(\omega) \\ \frac{x-\xi_1(\omega)}{\xi_2(\omega)-\xi_1(\omega)} & \text{if } \xi_1(\omega) \le x < \xi_2(\omega) \\ 1 & \text{if } \xi_2(\omega) \le x \le \xi_3(\omega) \\ \frac{\xi_4(\omega)-x}{\xi_4(\omega)-\xi_3(\omega)} & \text{if } \xi_3(\omega) < x \le \xi_4(\omega) \\ 0 & \text{if } x > \xi_4(\omega), \end{cases}$$

for random variables $\xi_1 \le \xi_2 \le \xi_3 \le \xi_4$, convergence in distribution is equivalent to convergence in distribution of $(\xi_1, \xi_2, \xi_3, \xi_4)$ as a random vector in $\mathbb{R}^4$.

Theorem 3.5. *[3, Theorem 2] Let* $p \in [1, \infty)$. *Let* X_n *be* $Tra(\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \xi_{n,4})$ *where* $\xi_{n,1} \le \xi_{n,2} \le \xi_{n,3} \le \xi_{n,4}$ *are random variables, and analogously* $X = Tra(\xi_1, \xi_2, \xi_3, \xi_4)$. *Then* $X_n \to X$ *in distribution in* d_p *if and only if, as random vectors in* $\mathbb{R}^4$, $(\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \xi_{n,4}) \to (\xi_1, \xi_2, \xi_3, \xi_4)$ *in distribution.*

That theorem relies on the following lemma ([3, Lemma 5]) whose proof is presented here (due to space reasons, it was omitted in [3]).

Lemma 3.6. *Let* $\{U_n\}_n$ *be a sequence of trapezoidal fuzzy numbers converging to some* $U \in \widehat{\mathcal{F}}_{c,1}(\mathbb{R})$ *in* d_1. *Then* $\{\|(U_n)_0\|\}_n$ *is bounded.*

Proof. Reasoning by contradiction, assume $\{\|(U_n)_0\|\}_n$ is not bounded. Since

$$\|(U_n)_0\| = \max\{|\inf(U_n)_0|, |\sup(U_n)_0|\}$$

we may assume without loss of generality, that $\{|\sup(U_n)_0|\}_n$ is not bounded, otherwise, replace U_n by $-U_n$. Then there exists a subsequence $\{U_{n'}\}_n$ such that for every $M > 0$ there exists some $N \in \mathbb{N}$ such that $\sup(U_{n'})_0 > M$ whenever $n' > N$. Moreover,

$$\begin{aligned} \|U_n\|_1 = d_1(U_n, I_{\{0\}}) = \int_{[0,1]} d_H((U_n)_\alpha, \{0\}) d\alpha \ge \int_{[0,1]} |\sup(U_n)_\alpha| d\alpha \\ = \int_{[0,1]} |\alpha \sup(U_n)_1 + (1-\alpha)\sup(U_n)_0| d\alpha \ge \int_{[0,1]} (\alpha \sup(U_n)_1 + (1-\alpha)\sup(U_n)_0) d\alpha \\ = \frac{1}{2}(\sup(U_n)_1 + \sup(U_n)_0). \end{aligned}$$

Now, for $V \in \widehat{\mathcal{F}}_{c,1}(\mathbb{R})$ we have

$$\|U_n\|_1 = d_1(U_n, I_{\{0\}}) \le d_1(U_n, V) + d_1(V, I_{\{0\}}) = d_1(U_n, U) + \|V\|_1.$$

Then

$$\|V\|_1 \ge \|U_n\|_1 - d_1(U_n, V) \ge \frac{1}{2}(\sup(U_n)_1 + \sup(U_n)_0) - d_1(U_n, V).$$

Set

$$M = \max\{1, 2\|U\|_1 - \inf_{n'}(\sup(U_{n'})_1) + 4)\}.$$

Then there exists some $N_1 \in \mathbb{N}$ such that $\sup(U_{n'})_0 > M$ for all $n > N_1$. Moreover, there exist $N_2 \in \mathbb{N}$ such that $d_1(U_n, U) < 1$ for all $n > N_2$. Let $N = \max\{N_1, N_2\}$. Then for some $n' > N$,

$$\|U\|_1 \geq \frac{1}{2}(\sup(U_{n'})_1 + \sup(U_{n'})_0) - d_1(U_{n'}, U) \geq \frac{1}{2}(\sup(U_{n'})_1 + M) - 1$$
$$\geq \frac{1}{2}(\sup(U_{n'})_1 + 2\|U\|_1 - \inf_n(\sup(U_{n'})_1) + 4) - 1 \geq \frac{1}{2}(+2\|U\|_1 + 4) - 1 = \|U\|_1 + 1,$$

where the last inequality uses the fact that

$$\sup(U_{n'})_1 - \inf_{n'} \sup((U_{n'})_1) \geq 0.$$

Since $\|U\|_1 \geq \|U\|_1 + 1$ is a contradiction, we conclude that $\{(U_n)_0\}_n$ is bounded. □

Remark 3.1. Since the space $(\widehat{\mathcal{F}}_{c,p}(\mathbb{R}^d), d_p)$ is complete, it is possible to assume that the limit of the sequence of trapezoidal fuzzy numbers belongs to $\widehat{\mathcal{F}}_{c,p}(\mathbb{R}^d)$.

As an application of Theorem 3.5, we obtained the following result relating convergence in distribution of random variables to convergence in distribution of their indicator functions ([3, Corollary 1]). It will be subsumed by Proposition 6.3 below which applies more generally to random sets.

Corollary 3.7. *Let ξ_n, ξ be random variables. Then $\xi_n \to \xi$ in distribution if and only if $I_{\{\xi_n\}} \to I_{\{\xi\}}$ in distribution in d_p.*

Lastly, we have [3, Proposition 1], which shows that convergence in distribution behaves nicely with respect to the arithmetic operations of fuzzy sets. More general results will be established in Section 5.

Proposition 3.8. *Let X_n, X be fuzzy random variables such that $X_n \to X$ in distribution in d_p. Then*

1. *For every $U \in \mathcal{F}_c(\mathbb{R}^d)$, we have $X_n + U \to X + U$ in distribution in d_p.*
2. *For every $a \in \mathbb{R}$, we have $aX_n \to aX$ in distribution in d_p.*

4. k-Tuples of fuzzy random variables

Since fuzzy random variables can take as values fuzzy subsets of $\mathbb{R}^d$ for $d > 1$, usually a clear distinction between a fuzzy random variable in $\mathbb{R}^d$ and a d-tuple of fuzzy random variables in $\mathbb{R}$ is not made. By this, we mean that a fuzzy random variable in $\mathbb{R}^d$ may just be called a fuzzy random vector (e.g., [9]).

In the non-fuzzy case, the space $\mathbb{R}^d$ and the Cartesian product of d copies of $\mathbb{R}$ are the same thing. But clearly, $\mathcal{F}_c(\mathbb{R}^2)$ is not the same thing as $\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R})$. At the intuitive level, each random element (X, Y) of $\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R})$ can be identified canonically with an element of $\mathcal{F}_c(\mathbb{R}^2)$ by taking the Cartesian product $X \times Y$. That is done, for instance, in [25, Section 5] where pairs of trapezoidal one-dimensional fuzzy data are modelled and graphically represented by using their 2-dimensional Cartesian product.

For a finite data sample, that raises no questions. For random elements, which may take infinitely many values, one can and should ask for a rigorous proof that the measurability of (X, Y) is the same thing as the measurability of $X \times Y$. And when convergence enters the picture, further questions need to be addressed as it is not self-evident that convergence of (X_n, Y_n) is the same thing as convergence of $X_n \times Y_n$. The aim of this section is to settle those questions satisfactorily.

To improve proof clarity, we will state the results in the way we will need to use them (for pairs of multidimensional fuzzy sets) but they clearly apply in the general case of k-tuples. We will merely state the case of k-tuples of one-dimensional fuzzy random variables which is not used in the sequel.

We will need the following theorem from the theory of standard probability spaces [19, Corollary 3.3, p. 22].

Lemma 4.1. *If B is a Borel subset of a complete separable metric space $\mathbb{E}$ and φ is an injective measurable map from B into a separable metric space $\mathbb{F}$, then $\varphi(B)$ is a Borel subset of $\mathbb{F}$, and φ is an isomorphism between the measurable spaces B and $\varphi(B)$ endowed with their Borel σ-algebras.*

The following lemma on the Hausdorff metric between products will be useful.

Lemma 4.2. *For every $K, K', L \in \mathcal{K}_c(\mathbb{R}^d)$ we have*

$$d_H(K \times L, K' \times L) = d_H(K, K').$$

Proof. Let $K, K', L \in \mathcal{K}_c(\mathbb{R}^d)$. Then

$$\begin{aligned}
&d_H(K \times L, K' \times L)\\
&= \max\{\sup_{(x,y)\in(K,L)} \inf_{(x',y')\in(K',L)} \|(x,y)-(x',y')\|, \sup_{(x',y')\in(K',L)} \inf_{(x,y)\in(K,L)} \|(x,y)-(x',y')\|\}\\
&= \max\{\sup_{x\in K, y\in L} \inf_{x'\in K', y'\in L} \|(x,y)-(x',y')\|, \sup_{x'\in K', y'\in L} \inf_{x\in K, y\in L} \|(x,y)-(x',y')\|\}\\
&= \max\{\sup_{x\in K, y\in L} \inf_{x'\in K', y'\in L} \|(x-x', y-y')\|, \sup_{x'\in K', y'\in L} \inf_{x\in K, y\in L} \|(x-x', y-y')\|\}\\
&= \max\{\sup_{x\in K, y\in L} \inf_{x'\in K', y'\in L} \|x-x'\|, \sup_{x'\in K', y'\in L} \inf_{x\in K, y\in L} \|x-x'\|\}\\
&= \max\{\sup_{x\in K,} \inf_{x'\in K'} \|x-x'\|, \sup_{x'\in K'} \inf_{x\in K} \|x-x'\|\} = d_H(K, K'). \quad \square
\end{aligned}$$

Consider the mapping

$$\begin{aligned}
\varphi_\times : (\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'}), d_{\max}) &\to (\mathcal{F}_c(\mathbb{R}^{d+d'}), d_p)\\
(U, V) &\to U \times V
\end{aligned}$$

where $d_{\max}(U \times V, U' \times V') = \max\{d_p(U, U'), d_p(V, V')\}$. Recall that the metric $d_{\max}$ induces the product topology in $\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'})$. The fact that $d_{\max}$ is chosen plays no role (any other equivalent metric generating the product topology would do) but the specifics of $d_{\max}$ will be used in the proofs.

Theorem 4.3. *The mapping $\varphi_\times$ is an homeomorphism onto its image and $\varphi_\times(\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'}))$ is a measurable subset of $\mathcal{F}_c(\mathbb{R}^{d+d'})$.*

Proof. (1) First, we have to see that $\varphi_\times$ is well defined. Let $U, V \in \mathcal{F}_c(\mathbb{R}^d)$. For every $x = (x_1, x_2) \in \mathbb{R}^{d+d'}$ (where x_1 has d components and x_2 has d' components),

$$(\varphi_\times(U, V))(x) = (U \times V)(x) = \min\{U(x_1), V(x_2)\} \in [0, 1].$$

By Lemma 2.7,

$$(\varphi_\times(U, V))_\alpha = (U \times V)_\alpha = U_\alpha \times V_\alpha$$

for each $\alpha \in [0, 1]$. Since U_α is non-empty, $U_\alpha \times V_\alpha$ is not-empty for any $\alpha \in [0, 1]$. Next, the product of compact convex sets in compact and convex.

(2) Let us show that $\varphi_\times$ is continuous. Let $(U_n, V_n) \to (U, V)$ in $d_{\max}$. First,

$$d_p(\varphi_\times(U_n, V_n), \varphi_\times(U, V)) \le d_p(\varphi_\times(U_n, V_n), \varphi_\times(U, V_n)) + d_p(\varphi_\times(U, V_n), \varphi_\times(U, V)).$$

Next, by Lemma 2.7 and Lemma 4.2,

$$\varphi_\times(U_n, V_n)_\alpha = (U_n)_\alpha \times (V_n)_\alpha.$$

Analogously, $\varphi_\times(U, V_n)_\alpha = U_\alpha \times (V_n)_\alpha$. Therefore, with Lemma 2.7,

$$d_H(\varphi_\times(U_n, V_n)_\alpha, \varphi_\times(U, V_n)_\alpha) = d_H((U_n)_\alpha \times (V_n)_\alpha, U_\alpha \times (V_n)_\alpha) = d_H((U_n)_\alpha, U_\alpha).$$

Thus

$$d_p(\varphi_\times(U_n, V_n), \varphi_\times(U, V_n)) = \left(\int_{[0,1]} d_H((U_n)_\alpha, U_\alpha)^p d\alpha \right)^{1/p} = d_p(U_n, U).$$

Analogously, $d_p(\varphi_\times(U, V_n), \varphi_\times(U, V)) = d_p(V_n, V)$. Then

$$d_p(\varphi_\times(U_n, V_n), \varphi_\times(U, V)) \le d_p(U_n, U) + d_p(V_n, V) \le 2d_{\max}((U_n, V_n), (U, V)) \to 0.$$

(3) We need to show that $\varphi_\times$ is an injective function. Let $W, W' \in \mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'})$, that is, $W = (U, V)$ and $W' = (U', V')$. Consequently, $\varphi_\times(W)_\alpha = U_\alpha \times V_\alpha$ and $\varphi_\times(W')_\alpha = U'_\alpha \times V'_\alpha$ for every $\alpha \in [0, 1]$. Provided $\varphi_\times(W) = \varphi_\times(W')$, then $U_\alpha = U'_\alpha$ and $V_\alpha = V'_\alpha$ for every α, hence $W = W'$.

(4) Denote by $\phi_\times$ the inverse function of $\varphi_\times$ defined by

$$\begin{aligned} \phi_\times : \varphi_\times(\mathcal{F}_c(\mathbb{R}^{d+d'})) &\to \mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'}) \\ W = U \times V &\to (U, V) \end{aligned}$$

Let $\{W_n\}_n = \{U_n \times V_n\}_n \subseteq \varphi_\times(\mathcal{F}_c(\mathbb{R}^{d+d'}))$ be a convergent sequence converging to $W = U \times V$.

$$d_{\max}(\phi_\times(W_n), \phi_\times(W)) = d_{\max}((U_n, V_n), (U, V)) = \max\{d_p(U_n, U), d_p(V_n, V)\}$$

with

$$d_p(U_n, U) = \left(\int_{[0,1]} [d_H((U_n)_\alpha, U_\alpha)]^p \, d\alpha \right)^{1/p}.$$

Then

$$\begin{aligned} & d_H((W_n)_\alpha, W_\alpha) = \\ & \max\{ \sup_{(x,y)\in(U_n)_\alpha\times(V_n)_\alpha} \inf_{(x',y')\in U_\alpha\times V_\alpha} \|(x, y) - (x', y')\|, \\ & \sup_{(x',y')\in U_\alpha\times V_\alpha} \inf_{(x,y)\in(U_n)_\alpha\times(V_n)_\alpha} \|(x, y) - (x', y')\|\}. \end{aligned}$$

Since all norms in a finite dimensional space are equivalent, there exists some constant $C > 0$ such that

$$\|(x, y) - (x', y')\| \ge C \cdot \max\{\|x - x'\|, \|y - y'\|\} \ge C \cdot \|x - x'\|$$

we have

$$\begin{aligned} & d_H((W_n)_\alpha, W_\alpha) \\ & \ge C \cdot \max(\sup_{(x,y)\in(U_n)_\alpha\times(V_n)_\alpha} \inf_{(x',y')\in U_\alpha\times V_\alpha} \|x - x'\|, \sup_{(x',y')\in U_\alpha\times V_\alpha} \inf_{(x,y)\in(U_n)_\alpha\times(V_n)_\alpha} \|x - x'\|) \\ & = C \cdot d_H((U_n)_\alpha, U_\alpha) \end{aligned}$$

Therefore $d_p(U_n, U) \le C^{-1} \cdot d_p(W_n, W)$. Analogously, $d_p(V_n, V) \le C^{-1} \cdot d_p(W_n, W)$. In conclusion,

$$d_{\max}(\phi_\times(W_n), \phi_\times(W)) \le C^{-1} \cdot d_p(W_n, W) \to 0.$$

(5) The measurability of $\varphi_\times(\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'}))$ is a consequence of Lemma 4.1. □

Remark 4.1. From the proof of Theorem 4.3, the inequalities

$$C \cdot d_{\max}(W, W') \le d_p(\varphi_\times(W), \varphi_\times(W')) \le 2 \cdot d_{\max}(W, W')$$

hold for all $W, W' \in \mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'})$. Hence $\varphi_\times$ is a bi-Lipschitz function.

With Theorem 4.3 to hand, it is not hard to answer the questions at the beginning of this section.

Proposition 4.4. *Let X, Y be mappings from a measurable space $(\Omega, \mathcal{A})$ to $\mathcal{F}_c(\mathbb{R}^d)$ and $\mathcal{F}_c(\mathbb{R}^{d'})$, respectively. Then the following are equivalent.*

(i) X *and* Y *are fuzzy random variables,*
(ii) (X, Y) *is a random element of the product space* $(\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'}), d_{\max})$*,*
(iii) $X \times Y$ *is a fuzzy random variable in* $\mathbb{R}^{d+d'}$*.*

Proof. For (i) ⇒ (ii), $(\mathcal{F}_c(\mathbb{R}^d), d_p)$ is a separable metric space, hence the product σ-algebra in $\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d)$ is $\mathcal{B}_{d_p} \otimes \mathcal{B}_{d_p}$. Therefore (X, Y) is measurable. For (ii) ⇒ (i), let $\pi_1 : \mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'}) \to \mathcal{F}_c(\mathbb{R}^d)$ be the projection over the first component of the product space. Then $X = \pi_1 \circ (X, Y)$. Since $\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'})$ is endowed with the product topology, π_1 is $(d_{\max}, d_p)$-continuous. Then, by Lemma 2.2, X is a fuzzy random variable in $\mathbb{R}^d$. Analogously, Y is a fuzzy random variable in $\mathbb{R}^{d'}$. Next, by Theorem 4.3, $\varphi_\times$ is an homeomorphism, therefore the mapping $X \times Y = \varphi_\times \circ (X, Y)$ is Borel measurable, giving (ii) ⇒ (iii). For (iii)⇒(ii), recall that $\phi_\times$ is also an homeomorphism by Theorem 4.3, hence $(X, Y) = \phi_\times \circ (X \times Y)$ is Borel measurable. □

An alternative form of the same ideas is as follows.

Proposition 4.5. *Let* $X_1, \ldots, X_k$ *be mappings from a measurable space* $(\Omega, \mathcal{A})$ *to* $\mathcal{F}_c(\mathbb{R})$*. Then the following are equivalent.*

(i) $X_1, \ldots, X_k$ *are fuzzy random variables,*
(ii) $(X_1, \ldots, X_k)$ *is a random element of the product space* $(\mathcal{F}_c(\mathbb{R}) \times \ldots \times \mathcal{F}_c(\mathbb{R}), d_{\max})$*, where*

$$d_{\max}((U_1, \ldots, U_n), (V_1, \ldots, V_n)) = \max_{i \in \{1,\ldots,n\}} d_p(U_i, V_i),$$

(iii) $X_1 \times \ldots \times X_k$ *is a fuzzy random variable in* $\mathbb{R}^k$*.*

As regards convergence in distribution, notice that $(X_n, Y_n) \to (X, Y)$ is not equivalent to X_n and Y_n converging separately to X and Y.

Proposition 4.6. *Let* X_n, X, Y_n, Y *be fuzzy random variables. Then* $(X_n, Y_n) \to (X, Y)$ *in distribution in* $d_{\max}$ *if and only if* $X_n \times Y_n \to X \times Y$ *in distribution in* d_p*.*

Proof. Assume $(X_n, Y_n) \to (X, Y)$ in distribution in $d_{\max}$. Let $h : \mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'}) \to \mathbb{R}$ be a continuous bounded mapping. Then

$$E[h(X_n \times Y_n)] = E[(h \circ \varphi_\times)(X_n, Y_n)] \to E[(h \circ \varphi_\times)(X, Y)] = E[h(X \times Y)],$$

since $h \circ \varphi_\times : \mathcal{F}_c(\mathbb{R}^{d+d'}) \to \mathbb{R}$ is continuous and bounded in d_p (by Theorem 4.3). Therefore $X_n \times Y_n \to X \times Y$ in distribution in $d_{\max}$.

For the converse, assume $X_n \times Y_n \to X \times Y$ in distribution in d_p and let G be an open set of $\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'})$. By Theorem 4.3, there exists an open set $\mathbf{G}$ of $\mathcal{F}_c(\mathbb{R}^{d+d'})$ such that $\varphi_\times(G) = \mathbf{G} \cap \varphi_\times(\mathcal{F}_c(\mathbb{R}^{d+d'}))$. Then

$$\begin{aligned}\liminf_{n\to\infty} P_{(X_n,Y_n)}(G) &= \liminf_{n\to\infty} (P_{X_n \times Y_n} \circ \varphi_\times)(G) = \liminf_{n\to\infty} P_{X_n \times Y_n}(\varphi_\times(G)) \\ &= \liminf_{n\to\infty} P_{X_n \times Y_n}(\mathbf{G} \cap \varphi_\times(\mathcal{F}_c(\mathbb{R}^{d+d'}))) = \liminf_{n\to\infty} P_{X_n \times Y_n}(\mathbf{G}).\end{aligned}$$

By Lemma 2.3,

$$\begin{aligned}\liminf_{n\to\infty} P_{X_n \times Y_n}(\mathbf{G}) &\geq P_{X \times Y}(\mathbf{G}) = P_{X \times Y}(\mathbf{G} \cap \varphi_\times(\mathcal{F}_c(\mathbb{R}^{d+d'}))) \\ &= P_{X \times Y}(\varphi_\times(G)) = (P_{X \times Y} \circ \varphi_\times)(G) = P_{(X,Y)}(G). \quad \square\end{aligned}$$

The analogous result in the context of Proposition 4.4 is as follows.

Proposition 4.7. *Let* $X_{n,1}, \ldots, X_{n,k}, X_1, \ldots, X_k$ *be fuzzy random variables in* $\mathbb{R}$*. Then* $(X_{n,1}, \ldots, X_{n,k}) \to (X_1, \ldots, X_k)$ *in distribution in the product topology of* $\prod_{i=1}^k (\mathcal{F}_c(\mathbb{R}), d_p)$ *if and only if* $X_{n,1} \times \ldots \times X_{n,k} \to X_1 \times \ldots \times X_k$ *in distribution in* $(\mathcal{F}_c(\mathbb{R}^k), d_p)$*.*

In the sequel, we will use Proposition 4.6 to recast joint convergence in distribution of fuzzy random variables (i.e., convergence of random elements of the product space $\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d)$) as a convergence of fuzzy random variables taking on values in $\mathcal{F}_c(\mathbb{R}^{2d})$ to which known results can directly be applied.

5. Slutski theorem

In this section, we will study the behaviour of convergence in distribution with respect to the operations of sum, product by a scalar and union (modelled using the maximum t-conorm). The union of two elements of $\mathcal{F}_c(\mathbb{R}^d)$ need not be in $\mathcal{F}_c(\mathbb{R}^d)$ but that will not be a problem since it is still in the sup-semilattice $\mathcal{F}(\mathbb{R}^d)$, where the d_p-metric is defined. There would actually be an essential problem to consider the intersection: its 1-cut can be empty whence the result can be out of the domain of the metric d_p.

We will prove results under joint convergence of the fuzzy random variables. In combination with a general result in [6] which states that separately convergent random elements do converge jointly provided one of the limits is degenerate, a generalization for fuzzy random variables of the Slutski theorem will follow. In this version we will consider the sum and product by scalars, but also the union operation which makes no sense for ordinary random variables.

Proposition 5.1. *Let $\{X_n\}_n$ and $\{Y_n\}_n$ be sequences of fuzzy random variables in $\mathbb{R}^d$ such that $\{(X_n, Y_n)\}_n$ converges in distribution to (X, Y) in $d_{\max}$. Then $X_n + Y_n$ converges in distribution in d_p to $X + Y$.*

Proof. Let $\varphi_\times$ be the embedding of $\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d)$ into $\mathcal{F}_c(\mathbb{R}^{2d})$. By Proposition 4.6, $X_n \times Y_n \to X \times Y$ in distribution in d_p. Next, define

$$s : (\varphi_\times(\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d)), d_p) \to (\mathcal{F}_c(\mathbb{R}^d), d_p)$$
$$U \times V \mapsto U + V$$

and

$$+ : (\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d), d_{\max}) \to (\mathcal{F}_c(\mathbb{R}^d), d_p)$$
$$(U, V) \mapsto U + V$$

Then s is (d_p, d_p)-continuous if and only if $+$ is $(d_{\max}, d_p)$-continuous. Let us check that s is a continuous function. Let $Z, T : (\{0,1\}, \{\{0,1\}, \{0\}, \{1\}, \emptyset\}, Q) \to \mathcal{F}_c(\mathbb{R}^d)$ be simple fuzzy random variables such that $Z(0) = A_1$, $Z(1) = A_2$, $T(0) = B_1$ and $T(1) = B_2$, with Q being the uniform distribution in $\{0, 1\}$. By [1, Lemma 4.4],

$$d_p(\frac{1}{2}A_1 + \frac{1}{2}A_2, \frac{1}{2}B_1 + \frac{1}{2}B_2) \le E[d_p(Z, T)]$$
$$= \int_{\{0,1\}} d_p(Z(\omega), T(\omega))dQ(\omega) = \frac{1}{2} \cdot [d_p(A_1, B_1) + d_p(A_2, B_2)].$$

Then

$$d_p(s(A_1 \times A_2), s(B_1 \times B_2)) = d_p(A_1 + A_2, B_1 + B_2) \le d_p(A_1, B_1) + d_p(A_2, B_2).$$

Therefore s is a continuous function. By Lemma 3.1, the sequence $\{X_n + Y_n\}_n = \{s(X_n, Y_n)\}_n$ converges in distribution to $X + Y$ in d_p. □

It is clear that the assumption of separate convergence of each sequence is not sufficient, as this is already the case for ordinary random variables and, by Corollary 3.7, one can identify a random variable ξ with the fuzzy random variable $I_{\{\xi\}}$.

We will consider the product by a scalar now. The proofs for the sum and union are similar and rely on identifying $\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d)$ with a subset of $\mathcal{F}_c(\mathbb{R}^{2d})$. In this case we need to consider $\mathbb{R} \times \mathcal{F}_c(\mathbb{R}^d)$. Although the problem can be handled to fit the same scheme (like in Proposition 6.5 below), we will take the opportunity to use a different proof method based on the Skorokhod theorem.

Theorem 5.2. *Let $\{X_n\}_n$ be a sequence of fuzzy random variables and let $\{\xi_n\}_n$ be a sequence of random variables such that $\{(\xi_n, X_n)\}_n$ converges in distribution to (ξ, X). Then $\xi_n \cdot X_n$ converges in distribution in d_p to $\xi \cdot X$.*

Proof. Let $Y_n = I_{\{\xi_n\}}$. Let $\varphi_\times$ be the embedding of $\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}^d)$ into $\mathcal{F}_c(\mathbb{R}^{d+1})$. By Proposition 4.6 we have $Y_n \times X_n \to Y \times X$ in distribution in d_p. By Lemma 3.2, there exist $Z_n, Z : ([0,1], \mathcal{B}_{[0,1]}, \ell) \to (\mathcal{F}_c(\mathbb{R}^{d+1}), d_p)$ distributed like $Y_n \times X_n$ and $Y \times X$, respectively, such that $Z_n \to Z$ almost sure. Furthermore, by Theorem 4.3, the set $\varphi_\times(\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}^d))$ is measurable. Then

$$\ell(Z_n \in \varphi_\times(\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}^d))) = P(\varphi_\times \circ (X_n, Y_n) \in \varphi_\times(\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}^d))) = 1.$$

Analogously,

$$\ell(Z \in \varphi_\times(\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}^d))) = 1.$$

Now, let N be a null measurable set which contains $\bigcup_{n\in\mathbb{N}}\{Z_n \notin \varphi_\times(\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}^d))\} \cup \{Z \notin \varphi_\times(\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}^d))\}$. Denote by π_1 the projection over the first component and π_2 the projection over the last d components. Next, define the following mappings:

$$Y'_n(t) = \begin{cases} (\pi_1 \circ \varphi_\times^{-1} \circ Z_n)(t) & \text{if } t \notin N \\ I_{\{0\}} & \text{if } t \in N \end{cases}$$

$$X'_n(t) = \begin{cases} (\pi_2 \circ \varphi_\times^{-1} \circ Z_n)(t) & \text{if } t \notin N \\ I_{\{0\}} & \text{if } t \in N \end{cases}$$

$$Y'(t) = \begin{cases} (\pi_1 \circ \varphi_\times^{-1} \circ Z)(t) & \text{if } t \notin N \\ I_{\{0\}} & \text{if } t \in N \end{cases}$$

$$X'(t) = \begin{cases} (\pi_2 \circ \varphi_\times^{-1} \circ Z)(t) & \text{if } t \notin N \\ I_{\{0\}} & \text{if } t \in N \end{cases}$$

Let us check that X' is a fuzzy random variable. For any $B \in \mathcal{B}_{\mathcal{F}_c(\mathbb{R}^d)}$,

$$(X')^{-1}(B) = (N \cap (X')^{-1}(B)) \cup (N^c \cap (X')^{-1}(B))$$
$$= \begin{cases} N \cup (N^c \cap (Z^{-1} \circ \varphi_\times \circ \pi_2^{-1})(B)) \in \mathcal{B}_{[0,1]} \text{ if } I_{\{0\}} \in B \\ \emptyset \cup (N^c \cap (Z^{-1} \circ \varphi_\times \circ \pi_2^{-1})(B)) \in \mathcal{B}_{[0,1]} \text{ if } I_{\{0\}} \notin B. \end{cases}$$

Analogously, X'_n, Y' and Y'_n are fuzzy random variables. Next, if $t \in N$, $X'_n(t) = I_{\{0\}} = X'(t)$. If $t \notin N$, $X'_n(t) = (\pi_1 \circ \varphi_\times^{-1} \circ Z_n)(t)$, $X'(t) = (\pi_1 \circ \varphi_\times^{-1} \circ Z_n)(t)$. By the continuity of $\pi_1 \circ \varphi_\times^{-1}$ and

$$d_p(X'_n(t), X'(t)) = d_p((\pi_1 \circ \varphi_\times^{-1} \circ Z_n)(t), (\pi_1 \circ \varphi_\times^{-1} \circ Z)(t)),$$

we have $X'_n(t) \to X'(t)$ for every $t \notin N$. Analogously, $Y'_n(t) \to Y'(t)$ for every $t \notin N$. Then (X'_n, Y'_n) converges almost surely to (X, Y) in $d_{\max}$.

Let us show that (Y'_n, X'_n) and (Y_n, X_n) have the same distribution. Let $(A_1, A_2) \in \mathcal{B}_{\mathcal{F}_c(\mathbb{R})} \otimes \mathcal{B}_{\mathcal{F}_c(\mathbb{R}^d)} = \mathcal{B}_{(\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}^d), d_{\max})}$.

$$\ell(\{t \in [0,1] : (Y'_n, X'_n)(t) \in (A_1, A_2)\}) = \ell(\{t \in N^c : (Y'_n, X'_n)(t) \in (A_1, A_2)\})$$
$$\ell(\{t \in N^c : (\varphi_\times^{-1} \circ Z_n)(t) \in (A_1, A_2)\}) = \ell(\{t \in [0,1] : (\varphi_\times^{-1} \circ Z_n)(t) \in (A_1, A_2)\})$$
$$\ell(\{t \in [0,1] : Z_n(t) \in \varphi_\times(A_1, A_2)\}) = P(\{\omega \in \Omega : (Y_n, X_n)(\omega) \in (A_1, A_2)\})$$

Then, by Proposition 4.4, $(Y'_n, X'_n), (Y', X') : ([0,1], \mathcal{B}_{[0,1]}, \ell) \to \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}^d)$ are fuzzy random variables such that $(Y'_n, X'_n) \to (Y', X')$ almost surely, $\ell_{(Y'_n, X'_n)} = P_{(Y_n, X_n)}$ and $\ell_{(Y', X')} = P_{(Y, X)}$.

Next, let us show that Y'_n is an indicator function almost surely. Let

$$i : \mathbb{R} \to \mathcal{F}_c(\mathbb{R})$$
$$a \mapsto I_{\{a\}}$$

By Lemma 4.1, $i(\mathbb{R})$ is Borel measurable. Then

$$\ell(Y' \in i(\mathbb{R})) = P(Y \in \mathbb{R}) = 1.$$

Analogously, $\ell(Y'_n \in i(\mathbb{R})) = 1$. Then let M be a null set containing $\bigcup_{n\in\mathbb{N}}\{Y'_n \notin i(\mathbb{R})\} \cup \{Y' \notin i(\mathbb{R})\}$. Consider the mappings

$$\eta_n(t) = \begin{cases} a & \text{if } Y'_n(t) = I_{\{a\}} \notin M \\ 0 & \text{if } t \in M \end{cases}$$

$$\eta(t) = \begin{cases} a & \text{if } Y'(t) = I_{\{a\}} \notin M \\ 0 & \text{if } t \in M \end{cases}$$

Let us show that (η_n, X'_n) has the same distribution as (ξ_n, X_n). Let $A \in \mathcal{B}_{\mathbb{R}\times\mathcal{F}_c(\mathbb{R}^d)}$. Denote by I_d the identity mapping in $\mathcal{F}_c(\mathbb{R}^d)$. Then

$$\begin{aligned}
&\ell(\{t \in [0,1] : (\eta_n, X'_n)(t) \in A\}) = \ell(\{t \in N^c \cap M^c : (\eta_n, X'_n)(t) \in A\}) \\
&= \ell(\{t \in N^c \cap M^c : (i, I_d)(\eta_n, X'_n)(t) \in (i, I_d)(A)\}) \\
&= \ell(\{t \in N^c \cap M^c : (Y'_n, X'_n)(t) \in (i, I_d)(A)\}) \\
&= \ell(\{t \in [0,1] : (Y'_n, X'_n)(t) \in (i, I_d)(A)\}) = P(\{\omega \in \Omega : (Y_n, X_n)(\omega) \in (i, I_d)(A)\}) \\
&= P(\{\omega \in \Omega : (\xi_n, X_n)(\omega) \in A\}).
\end{aligned}$$

Finally, let us show that $\{\eta_n \cdot X'_n\}_n$ converges almost sure to $\eta \cdot X'$, that is, $d_p(\eta_n \cdot X'_n(t), \eta \cdot X'(t)) \to 0$ for every $t \notin N \cup M$.

$$d_p((\eta_n \cdot X'_n)(t), (\eta \cdot X')(t)) \le d_p((\eta_n \cdot X'_n)(t), (\eta_n \cdot X')(t)) + d_p((\eta_n \cdot X')(t), (\eta \cdot X')(t)).$$

On the one hand,

$$d_p((\eta_n \cdot X'_n)(t), (\eta_n \cdot X')(t)) = |\eta_n(t)| d_p(X'_n(t), X'(t)) \to 0,$$

since $|\eta_n(t)|$ is bounded, being a convergent sequence to $|\eta(t)|$. On the other hand, by Lemma 2.1 we have

$$d_p(\eta_n \cdot Y, \eta \cdot Y)(t) = \left(\int_{[0,1]} [d_H([\eta_n \cdot Y]_\alpha, [\eta \cdot Y]_\alpha)]^p \, d\alpha \right)^{1/p} (t) =$$

$$= \left(\int_{[0,1]} [d_H([\eta_n(t) \cdot Y(t)]_\alpha, [\eta(t) \cdot Y(t)]_\alpha)]^p \, d\alpha \right)^{1/p}$$

$$\le |\eta_n(t) - \eta(t)| \left(\int_{[0,1]} \|Y_\alpha(t)\|^p \right)^{1/p} = |\eta_n(t) - \eta(t)| \|Y(t)\|_p \to 0,$$

since the $\eta_n(t)$ converges to $\eta(t)$ and $\|Y(t)\|_p < \infty$.

Then $d_p(\eta_n(t) \cdot X'_n(t), \eta(t) \cdot X'(t)) \to 0$ for each $t \notin N \cup M$. This proves that the function ϕ given by

$$\begin{aligned}
\phi : (\mathbb{R} \times \mathcal{F}_c(\mathbb{R}^d), d_{\max}) &\to (\mathcal{F}_c(\mathbb{R}^d), d_p) \\
(a, U) &\mapsto aU.
\end{aligned}$$

is continuous. Finally, it remains to show that $P_{\xi_n X_n} = \ell_{\eta_n X'_n}$. Let $A \in \mathcal{B}_{\mathcal{F}_c(\mathbb{R}^d)}$. Then

$$\begin{aligned}
&P(\{\omega \in \Omega : \xi_n(\omega) \cdot X_n(\omega) \in A\}) = P(\{\omega \in \Omega : (\xi_n, X_n)(\omega) \in \phi^{-1}(A)\}) \\
&= \ell(\{t \in [0,1] : (\eta_n, X'_n)(t) \in \phi^{-1}(A)\}) = \ell(\{t \in [0,1] : \eta_n(t) \cdot X'_n(t) \in A\}).
\end{aligned}$$

Analogously, $P_{\xi \cdot X} = \ell_{\eta \cdot Y}$. Since $\eta_n \cdot X'_n \to \eta \cdot X'$ almost surely, by the identical distribution we have $\xi_n \cdot X_n \to \xi \cdot X$ in distribution in d_p. □

As regards the union operation, we begin by establishing its continuity with respect to the relevant metrics. Set

$$\phi_\cup : (\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d), d_{\max}) \to (\mathcal{F}(\mathbb{R}^d), d_p)$$
$$(U, V) \mapsto U \cup V.$$

Lemma 5.3. *The function $\phi_\cup$ is continuous.*

Proof. Let $\{U_n\}_n$, $\{V_n\}_n$ be sequences of fuzzy sets which converge to U and V respectively in d_p.

$$d_p(U_n \cup V_n, U \cup V_n) = \left(\int_{[0,1]} [d_H((U_n \cup V_n)_\alpha, (U \cup V_n)_\alpha)]^p \, d\alpha \right)^{1/p}.$$

By Lemma 2.8, for each $\alpha \in [0, 1]$

$$d_H((U_n \cup V_n)_\alpha, (U \cup V_n)_\alpha) = d_H((U_n)_\alpha \cup (V_n)_\alpha), (U_\alpha \cup (V_n)_\alpha)) \leq d_H((U_n)_\alpha, U_\alpha).$$

Then

$$d_p(U_n \cup V_n, U \cup V_n) = \left(\int_{[0,1]} [d_H((U_n)_\alpha, U_\alpha)]^p \, d\alpha \right)^{1/p} = d_p(U_n, U).$$

Analogously,

$$d_p(U \cup V_n, U \cup V) = \left(\int_{[0,1]} [d_H((U \cup V_n)_\alpha, (U \cup V)_\alpha)]^p \, d\alpha \right)^{1/p}.$$

For each $\alpha \in [0, 1]$,

$$d_H((U \cup V_n)_\alpha, (U \cup V)_\alpha) = d_H((U_\alpha \cup (V_n)_\alpha), (U_\alpha \cup V_\alpha)) \leq d_H((V_n)_\alpha, V_\alpha).$$

Then

$$d_p(U \cup V_n, U \cup V) \leq \left(\int_{[0,1]} [d_H((V_n)_\alpha, V_\alpha)]^p \, d\alpha \right)^{1/p} = d_p(V_n, V).$$

With the triangle inequality,

$$d_p(U_n \cup V_n, U \cup V) \leq d_p(U_n, U) + d_p(V_n, V) \to 0. \quad \square$$

Like for the sum, we deduce that taking unions preserves convergence of sequences, provided the terms in the union converge jointly.

Proposition 5.4. *Let $\{X_n\}_n$ and $\{Y_n\}_n$ be sequences of fuzzy random variables in $\mathbb{R}^d$ such that $\{(X_n, Y_n)\}_n$ converges in distribution to (X, Y) in $d_{\max}$. Then $X_n \cup Y_n$ converges in distribution in d_p to $X \cup Y$.*

Proof. First, $U \cup V = \phi_\cup(U, V) = (\phi_\cup \circ \phi_\times)(U \times V)$. Since both $\phi_\cup$ and $\phi_\times$ are continuous (Lemma 5.3 and Theorem 4.3), their composition is continuous. Moreover, $X_n \times Y_n$ converges in distribution to $X \times Y$ in d_p (Lemma 4.6). Hence, by Lemma 3.1, the sequence $\{X_n \cup Y_n\}_n = \{(\phi_\cup \circ \phi_\times)(X_n, Y_n)\}_n$ converges in distribution to $X \cup Y$ in d_p. $\square$

To derive the Slutski theorem, we use a general result from [6, Exercise 3.10, pp. 49–50] (see also [21]).

Lemma 5.5. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of Radon probability measures on the product $\Omega_1 \times \Omega_2$ of two Hausdorff spaces and denote by $\{\mu_n\}_{n\in\mathbb{N}}$, respectively $\{\nu_n\}_{n\in\mathbb{N}}$ the marginal distribution of P_n on Ω_1, respectively Ω_2. If $\{\mu_n\}_{n\in\mathbb{N}}$ converges weakly to μ and $\{\nu_n\}_{n\in\mathbb{N}}$ converges weakly to some one-point measure c, then $\{P_n\}_{n\in\mathbb{N}}$ converges weakly to $\mu \otimes c$.*

Its application to our setting is as follows.

Lemma 5.6. *Let $\{X_n\}_n$ and $\{Y_n\}_n$ be sequences of fuzzy random variables, let X be a fuzzy random variable and $U \in \mathcal{F}_c(\mathbb{R}^d)$. If $X_n \to X$ and $Y_n \to U$ in distribution in d_p, then $(X_n, Y_n) \to (X, U)$ as random elements of $\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d)$.*

Proof. Since every Suslin space is Radon [23, Theorem 10, p. 122] and $(\mathcal{F}_c(\mathbb{R}^d), d_p)$ is a Lusin space for every $p \in [1, \infty)$ (see [1, Proposition 5.4]), which is a stronger condition that being a Suslin space, every probability measure in $(\mathcal{F}_c(\mathbb{R}^d), d_p)$ is a Radon measure. By Proposition 2.2, those probability measures are exactly those induced by fuzzy random variables. Then Lemma 5.5 can be applied with $\Omega_1 = \Omega_2 = \mathcal{F}_c(\mathbb{R}^d)$, $\mu_n = P_{X_n}$ and $\eta_n = P_{Y_n}$, whence $(X_n, Y_n) \to (X, U)$ in distribution in d_p. □

We finally obtain the Slutski theorem for fuzzy random variables under the d_p metrics.

Corollary 5.7. *Let $\{X_n\}_n$ be a sequence of fuzzy random variables which converges in distribution to X in d_p, let $\{Y_n\}_n$ be a sequence of fuzzy random variables converging in distribution to a constant U and let $\{\xi_n\}_n$ be a sequence of random variables converging in distribution to ξ.*

(a) *If ξ is a degenerate random variable which takes on the value c, then $\xi_n \cdot X_n$ converges in distribution in d_p to $c \cdot X$.*
(b) *If X is a degenerate fuzzy random variable which takes on the value U, then $\xi_n \cdot X_n$ converges in distribution in d_p to $\xi \cdot U$.*
(c) *$X_n + Y_n$ converges in distribution in d_p to $X + U$.*
(d) *$X_n \cup Y_n$ converges in distribution in d_p to $X \cup U$.*

Proof. For part (a), replacing Ω_1 by $\mathcal{F}_c(\mathbb{R}^d)$, Ω_2 by $\mathbb{R}^d$, μ_n by P_{X_n} and η_n by P_{ξ_n} in Lemma 5.5, we have $(\xi_n, X_n) \to (c, X)$ in distribution and by Theorem 5.2 it holds that $\xi_n \cdot X_n \to c \cdot X$ in distribution in d_p. Part (b) is analogous. For part (c), we have to apply Lemma 5.6 and Proposition 5.1, giving $X_n + Y_n \to X + U$. Similarly, part (d) is a combination of Proposition 5.4 and Lemma 5.6. □

6. Further properties

This section collects a brief study of other properties of convergence in distribution, concerning the inclusion ordering, random sets, and d_p-continuous transformations in $\mathcal{F}_c(\mathbb{R}^d)$.

We will first show that if all possible values taken by a convergence sequence are 'covered' by a fixed element of $\mathcal{F}_c(\mathbb{R}^d)$ then the same property holds for its limit in distribution. Notice that the result is stated for d_1 because it is weaker than any other d_p-metric.

Let $K \in \mathcal{K}_c(\mathbb{R}^d)$ and let $x \in \mathbb{R}^d$. Denote by d the distance function

$$d(x, K) = \inf_{y \in K} \|x - y\|.$$

Proposition 6.1. *Let $\{X_n\}_n$ be a sequence of fuzzy random variables which converges in distribution in d_1 to X. If $X_n \subseteq U$ for some $U \in \mathcal{F}_c(\mathbb{R}^d)$ and for each $n \in \mathbb{N}$, then $X \subseteq U$ almost surely.*

Proof. Set

$$\mathfrak{U} = \{V \in \mathcal{F}_c(\mathbb{R}^d) : V \subseteq U\}.$$

Let us show that $\mathfrak{A}$ is closed in $(\mathcal{F}_c(\mathbb{R}^d), d_1)$. Let $\{V_n\}_n \subseteq \mathfrak{A}$ be a sequence of fuzzy sets which converges to some V in d_1, that is, $d_1(V_n, V) \to 0$. By Corollary 2.6 there exists some $N \subseteq [0, 1]$ such that $\ell(N) = 1$ and $(V_n)_\alpha \to V_\alpha$ for all $\alpha \in N$. We claim that $V_\alpha \subseteq U_\alpha$. For, if $x_0 \notin U_\alpha$ for some $x_0 \in V_\alpha$, then there would exist $\varepsilon > 0$ such that $d(x_0, U_\alpha) > \varepsilon$, and then

$$d_H(V_\alpha, (V_n)_\alpha) \geq \sup_{x \in V_\alpha} d(x, (V_n)_\alpha) \geq d(x_0, (V_n)_\alpha) \geq d(x_0, U_\alpha) > \varepsilon,$$

a contradiction. In order to extend the inclusion $V_\alpha \subseteq U_\alpha$ to $\alpha \notin N$, consider now any $\alpha \in (0, 1] \setminus N$. Let us show $V_\alpha = \bigcap_{N \cap \{\beta < \alpha\}} V_\beta$.

($\subseteq$) $V_\alpha = \bigcap_{\{\beta < \alpha\}} V_\beta \subseteq \bigcap_{N \cap \{\beta < \alpha\}} V_\beta$.

($\supseteq$) Let $x \notin \bigcap_{\{\beta < \alpha\}} V_\beta$. Then there exists $\beta^* < \alpha$ such that $x \notin V_\alpha$ for all $\alpha > \beta^*$. Since $\ell(N) = 1$, N is a dense subset of $[0, 1]$. Therefore there exists $\alpha^* \in N \cap (\beta^*, \alpha)$ such that $x \notin V_{\alpha^*}$. Then $x \notin \bigcap_{N \cap \{\beta < \alpha\}} V_\beta$. Analogously, $U_\alpha = \bigcap_{N \cap \{\beta < \alpha\}} U_\beta$. Therefore

$$V_\alpha = \bigcap_{N \cap \{\beta < \alpha\}} V_\beta \subseteq \bigcap_{N \cap \{\beta < \alpha\}} U_\beta = U_\alpha.$$

There only remains the case $\alpha = 0 \notin N$, that is, $d_H(V_0, (V_n)_0)$ does not converge to 0. Let us show $V_0 = \operatorname{cl} \bigcup_{\{\beta > 0\} \cap N} V_\beta$.

($\subseteq$) Suppose that there exists $x \in V_0$ such that $x \notin \operatorname{cl} \bigcup_{\{\beta > 0\} \cap N} V_\beta$. Then for every $\beta \in N$, $\beta > 0$ we have $x \notin V_\beta$. Therefore $x \in V_{\beta^*}$ with $\beta^* > 0$, $\beta^* \notin N$. This contradicts the fact that $V_{\beta^*} = \bigcap_{N \cap \{\beta < \beta^*\}} V_\beta$.

($\supseteq$) By definition, $V_0 = \operatorname{cl} \bigcup_{\{\beta > 0\}} V_\beta \supseteq \operatorname{cl} \bigcup_{\{\beta > 0\} \cap N} V_\beta$.

Then

$$V_0 = \operatorname{cl} \bigcup_{\{\beta > 0\} \cap N} V_\beta \subseteq \operatorname{cl} \bigcup_{\{\beta > 0\} \cap N} U_\beta \subseteq U_0.$$

Since $V_\alpha \subseteq U_\alpha$ for all $\alpha \in [0, 1]$, we have $V \subseteq U$ proving that $\mathfrak{A}$ is closed. By the portmanteau theorem (Lemma 2.3),

$$1 = P(X_n \subseteq U) = P(X_n \in \mathfrak{A}) \leq \limsup_n P_{X_n}(\mathfrak{A}) \leq P_X(\mathfrak{A}) = P(X \in \mathfrak{A}) = P(X \subseteq U)$$

whence $X \subseteq U$ almost surely. □

Example 6.1. The previous result is not true if we replace U with a fuzzy random variable Y. Suppose that $Z \sim N(0, 1)$ and $X_n, Y = I_{\{Z\}}$, $X = I_{\{-Z\}}$. Then $X_n \subseteq Y$ for each $n \in \mathbb{N}$ and $X_n \to X$ in distribution in d_1, but $P(X \subseteq Y) = P(Z = 0) = 0$.

Proposition 6.1 raises the question whether the deterministic fuzzy set U can be replaced by a random Y provided the convergence is suitably strengthened. The answer is positive, using convergence in probability.

Proposition 6.2. *Let X_n, Y_n, X, Y be fuzzy random variables such that $X_n \to X$ and $Y_n \to Y$ in probability in d_1. Suppose that $X_n \subseteq Y_n$ for all $n \in \mathbb{N}$. Then $X \subseteq Y$ almost surely.*

Proof. Since $X_n \to X$ in probability, by [14, Lemma 3.2] there exists a subsequence $\{X_{n'}\}_n$ of $\{X_n\}_n$ which converges to X almost surely in d_1. By the same reasoning, there exists a further subsequence $\{Y_{n''}\}_n$ of $\{Y_{n'}\}_n$ which converges to Y almost surely in d_1. Then also $X_{n''} \to X$ almost sure in d_1.

Let $\omega \in \Omega$ be such that $X_n(\omega) \to X(\omega)$ and $Y_n(\omega) \to Y(\omega)$. Then, by Corollary 2.6 there exists a subset N_1 of $[0, 1]$ with $\ell(N_1) = 1$ such that $X_n(\omega)_\alpha \to X(\omega)_\alpha$ in d_H for every $\alpha \in N_1$. Analogously, there exists a set $N_2 \subseteq [0, 1]$ with $\ell(N_2) = 1$ such that $Y_n(\omega)_\alpha \to Y(\omega)_\alpha$ in d_H for every $\alpha \in N_2$. Then for every $\alpha \in N = N_1 \cap N_2$ we have $d_H(X_n(\omega)_\alpha, X(\omega)_\alpha) \to 0$ and $d_H(Y_n(\omega)_\alpha, Y(\omega)_\alpha) \to 0$.

Let $\alpha \in N$. Although N depends on ω, this will not be an obstacle for the proof. For any fixed $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $X(\omega)_\alpha \subseteq X_n(\omega)_\alpha + \varepsilon B$ and $X_n(\omega)_\alpha \subseteq X(\omega)_\alpha + \varepsilon B$ for every $n \geq n_1$. Analogously, there exists $n_2 \in \mathbb{N}$ such that $Y(\omega)_\alpha \subseteq Y_n(\omega)_\alpha + \varepsilon B$ and $Y_n(\omega)_\alpha \subseteq Y(\omega)_\alpha + \varepsilon B$ for every $n \geq n_2$. Furthermore, $X_n(\omega)_\alpha \subseteq Y_n(\omega)_\alpha$ for all $n \in \mathbb{N}$. Let $n^* = \max\{n_1, n_2\}$. Then

$$X(\omega)_\alpha \subseteq X_n(\omega)_\alpha + \varepsilon B \subseteq Y_n(\omega)_\alpha + \varepsilon B \subseteq Y(\omega)_\alpha + \varepsilon B + \varepsilon B$$

for every $n \geq n^*$. By the arbitariness of ε, $X_\alpha(\omega) \subseteq Y_\alpha(\omega)$ for every $\alpha \in N_1 \cap N_2$. Reasoning like in the proof of Proposition 6.1, we obtain $X_\alpha(\omega) \subseteq Y_\alpha(\omega)$ for every $\alpha \in [0,1]$. Therefore $X(\omega) \subseteq Y(\omega)$. In conclusion, $X \subseteq Y$ almost surely. □

Another natural question is whether the identification of a set with its indicator function is respected by convergence in distribution. The first part of the answer is as follows.

Proposition 6.3. *Let $X_n, X : (\Omega, \mathcal{A}, P) \to (\mathcal{K}_c(\mathbb{R}^d), d_H)$ be random sets. Then $X_n \to X$ in distribution in d_H if and only if $I_{X_n} \to I_X$ in distribution in d_p for any $p \in [1, \infty)$.*

Proof. ($\Rightarrow$) Suppose that $X_n \to X$ in distribution in d_H and let $p \in [1, \infty)$. By the Skorokhod representation theorem [26], there exist random sets $Y_n, Y : ([0,1], \mathcal{B}_{[0,1]}, \ell) \to (\mathcal{K}_c(\mathbb{R}^d), d_H)$ such that $\ell_{Y_n} = P_{X_n}$, $\ell_Y = P_X$ and $Y_n(t) \to Y(t)$ for each $t \in [0,1]$. Since

$$d_p(I_{Y_n}(t), I_Y(t)) = d_H(Y_n(t), Y(t)),$$

it follows that $I_{Y_n}(t) \to I_Y(t)$ in d_p for each t. Let

$$\begin{aligned} i : (\mathcal{K}_c(\mathbb{R}^d), d_H) &\to (\mathcal{F}_c(\mathbb{R}^d), d_p) \\ U &\mapsto I_U. \end{aligned}$$

For any $A \in \mathcal{B}_{(\mathcal{F}_c(\mathbb{R}^d), d_p)}$,

$$\begin{aligned} &\ell_{I_{Y_n}}(A) = \ell(\{t \in [0,1] : I_{Y_n}(t) \in A\}) = \ell(\{t \in [0,1] : Y_n(t) \in i^{-1}(A)\}) \\ &= P(\{\omega \in \Omega : X_n(\omega) \in i^{-1}(A)\}) = P(\{\omega \in \Omega : I_{X_n}(\omega) \in A\}) = P_{I_{X_n}}(A). \end{aligned}$$

Analogously, $\ell_{I_Y} = P_{I_X}$. Since I_{X_n}, I_X are identically distributed as I_{Y_n}, I_Y, therefore $I_{X_n} \to I_X$ in distribution in d_p.

($\Leftarrow$) Suppose that $I_{X_n} \to I_X$ in distribution in d_p. By Theorem 3.2, there exist fuzzy random variables $Y_n, Y : ([0,1], \mathcal{B}_{[0,1]}, \ell) \to (\mathcal{F}_c(\mathbb{R}^d), d_p)$ such that $\ell_{Y_n} = P_{I_{X_n}}$, $\ell_Y = P_{I_X}$ and $Y_n(t) \to Y(t)$ for each $t \in [0,1]$. For any $A \in \mathcal{B}_{(\mathcal{K}_c(\mathbb{R}^d), d_H)}$,

$$\begin{aligned} &P(\{\omega \in \Omega : X_n(\omega) \in A\}) = P(\{\omega \in \Omega : I_{X_n}(\omega) \in i(A)\}) \\ &= \ell(\{t \in [0,1] : Y_n(t) \in i(A)\}) = \ell(\{t \in [0,1] : i^{-1} \circ Y_n(t) \in A\}). \end{aligned}$$

Since i is an isometry, $d_p((i^{-1} \circ Y_n)(t), (i^{-1} \circ Y)(t)) = d_H(Y_n(t), Y(t))$. In conclusion, $X_n \to X$ in distribution in d_H. □

Proposition 6.3 falls short of answering the question satisfactorily: in the literature of random sets, it is much more common to define convergence in distribution using the weaker Fell topology than the Hausdorff metric (see, e.g., [17, Definition 6.1, p. 84–85]). Let us reason that both are equivalent in our context.

Corollary 6.4. *Let $X_n, X : (\Omega, \mathcal{A}, P) \to \mathcal{K}_c(\mathbb{R}^d)$ be random sets. Then $X_n \to X$ in distribution in the Fell topology if and only if $I_{X_n} \to I_X$ in distribution in d_p for every $p \in [1, \infty)$.*

Proof. By [22, Corollary 3A], convergence of compact convex sets in d_H is equivalent to their convergence in the Fell topology. Since the latter is metrizable (see Theorem 2.2 in [29], where Fell topology is called hit-or-miss topology), the equivalence of the convergences implies that the topologies are identical. Since the definition of weak convergence depends on the topology only, convergence in distribution with respect to d_H is the same thing as convergence in distribution with respect to τ_F, for random sets with compact convex values. Thus Proposition 6.3 yields the result. □

Lastly, let us show that the role of $\mathbb{R}$ as the codomain of the mappings in the definition of convergence in distribution can be played also by $\mathcal{F}_c(\mathbb{R}^d)$ itself, resulting in an equivalent definition.

Proposition 6.5. *Let X_n, X be fuzzy random variables. Then $X_n \to X$ in distribution in d_p if and only if for every continuous and bounded function $\varphi : (\mathcal{F}_c(\mathbb{R}^d), d_p) \to (\mathcal{F}_c(\mathbb{R}^d), d_p)$ we have $E[\varphi(X_n)] \to E[\varphi(X)]$.*

Proof. *Necessity:* By the continuous mapping theorem (Lemma 3.1), $\varphi(X_n) \to \varphi(X)$ in distribution in d_p. Since φ is bounded, there exists $R > 0$ such that

$$d_p(\varphi(X_n), I_{\{0\}}) < R,$$

hence there exists an integrable function $g : \Omega \to \mathbb{R}$ (given by $g(\omega) = R$ for each ω) such that $d_p(\varphi(X_n), I_{\{0\}}) \le g$. Finally, by the dominated convergence theorem (Lemma 3.3), $E[\varphi(X_n)] \to E[\varphi(X)]$.

Sufficiency: Let us show that $E[f(X_n)] \to E[f(X)]$ for every continuous bounded function $f : (\mathcal{F}_c(\mathbb{R}^d), d_p) \to \mathbb{R}$. Let $f : (\mathcal{F}_c(\mathbb{R}^d), d_p) \to \mathbb{R}$ be any continuous bounded function. Let $U_n, U \in \mathcal{F}_c(\mathbb{R}^d)$ such that $U_n \to U$. Then

$$\begin{aligned} &d_p(I_{\{(f(U_n),0,\ldots,0)\}}, I_{\{(f(U),0,\ldots,0)\}}) = d_H(\{(f(U_n), 0, \ldots, 0)\}, \{(f(U), 0, \ldots, 0)\}) \\ &= |f(U_n) - f(U)|, \end{aligned}$$

hence $I_{\{(f(\cdot),0,\ldots,0)\}} : (\mathcal{F}_c(\mathbb{R}^d), d_p) \to \mathbb{R}$ is d_p-continuous and bounded. Then $E[I_{\{f(X_n),0,\ldots,0\}}] \to E[I_{\{f(X),0,\ldots,0\}}]$ in d_p. Since

$$\begin{aligned} &d_p(E[I_{\{(f(X_n),0,\ldots,0)\}}], E[I_{\{(f(X),0,\ldots,0)\}}]) = d_p(I_{E_A[\{(f(X_n),0,\ldots,0)\}]}, I_{E_A[\{(f(X),0,\ldots,0)\}]}) \\ &= d_H(E_A[\{(f(X_n), 0, \ldots, 0)\}], E_A[\{(f(X), 0, \ldots, 0)\}]) \\ &= d_H(\{(E[f(X_n)], 0 \ldots, 0)\}, \{(E[f(X)], 0 \ldots, 0)\}) \\ &= \|(E[f(X_n)], 0 \ldots, 0) - (E[f(X_n)], 0 \ldots, 0)\| = \|(E[f(X_n)] - E[f(X)], 0 \ldots, 0)\| \\ &= |E[f(X_n)] - E[f(X)]|, \end{aligned}$$

indeed $E[f(X_n)] \to E[f(X)]$. □

7. A statistical application

This section is based on the fact that the cumulative distribution function of a bounded random variable can be identified with an element of $\mathcal{F}_c(\mathbb{R})$. It has been repeatedly observed that the left and right slopes of a fuzzy interval can be easily transformed into cumulative distribution functions (see, e.g., the Goetschel–Voxman representation theorem [10, Theorem 1.1]). This has sometimes been used to try to give fuzzy intervals a probabilistic interpretation. The novelty in this section is that we will show how to use results about $\mathcal{F}_c(\mathbb{R})$ in order to obtain results about probability distributions via their cumulative distribution functions. This is applied to illustrate the results in the preceeding sections but other applications are possible.

Fix an interval $[a, b]$. Let $\mathcal{P}_{[a,b]}$ be the set of all probability distributions whose support is contained in $[a, b]$. For any $p \in [1, \infty)$, the *L^p-Wasserstein distance* (see, e.g., [18]) between probability distributions $P, Q \in \mathcal{P}_{[a,b]}$ is

$$w_p(P, Q) = \inf_{X,Y} \|X - Y\|_p$$

where X and Y respectively have distribution P and Q.

For any $\alpha \in (0, 1]$, the *α-quantile* of P is

$$q(P, \alpha) = \inf\{x \in \mathbb{R} \mid P((-\infty, x]) \ge \alpha\}.$$

Since P has bounded support, $q(P, 1)$ is finite.

Vincentiles [28] were developed in the context of quantile estimation from samples from several groups. It was observed that pooling all the data from the different groups typically led to a multimodal distribution and it was preferred to estimate a quantile by averaging the quantiles of the different groups rather than by the corresponding quantile of the pooled data. This is often used in response time studies (e.g., [12,4]).

Let us present an abstract description of vincentiles. The *vincentized distribution* $\mathrm{Vin}(P, Q)$ of any P and Q is defined by the identities

$$q(\mathrm{Vin}(P,Q),\alpha)=\frac{q(P,\alpha)+q(Q,\alpha)}{2}$$

for $\alpha \in (0,1]$, yielding the cumulative distribution function

$$F_{\mathrm{Vin}}(x)=\sup\{\alpha \in (0,1] \mid q(\mathrm{Vin}(P,Q),\alpha)\geq x\}.$$

When P and Q are empirical distributions, we obtain the definition of the sample vincentile. Therefore, an α-vincentile is just the α-quantile of the vincentized distribution. While an extension to more subsamples is obvious, for notational simplicity we will develop the two-sample case.

Let P be the true probability distribution to be estimated. An *estimator* $\widehat{P}_n : \omega \in \Omega \mapsto \widehat{P}_n(\omega) \in \mathcal{P}_{[a,b]}$ of P is a random element of $\mathcal{P}_{[a,b]}$ which is based on the information of a sample of size n. Hence it has the form $T_n(\xi_1,\ldots,\xi_n)$ where $T_n : \mathbb{R}^n \to \mathcal{P}_{[a,b]}$ and $\{\xi_n\}_n$ is a sequence of random variables with distribution P (independent or otherwise).

While this description of estimation is nonparametric in nature, parametric problems can be rewritten in this form provided the dependence of the distribution on the parameter is measurable. It is also important to note that $\mathcal{P}_{[a,b]}$ is compact with the weak topology of probability measures and the L^p-Wasserstein metrics [18, Corollary 2.2.5]. They all generate the same topology and Borel σ-algebra, making the expression 'a random element of $\mathcal{P}_{[a,b]}$' univocal in meaning.

An estimator is *weakly consistent* if $\widehat{P}_n \to P$ in probability, equivalently if it converges weakly (since the limit is a deterministic element of $\mathcal{P}_{[a,b]}$).

We define a *generalized vincentile function* by

$$v_{n,\cdot}(\alpha)=q(\mathrm{Vin}(\widehat{P}_n,\widehat{Q}_n),\alpha), \quad \alpha \in (0,1]$$

where $\widehat{P}_n$, $\widehat{Q}_n$ are estimators of P. The vincentized distribution $\mathrm{Vin}(\widehat{P}_n,\widehat{Q}_n)$ will be called the *vincentized estimator*.

We will connect all this to fuzzy intervals using the following results.

Lemma 7.1. (See [18, p. 48–49]) *Let P, Q be probability distributions in $\mathbb{R}$. Then*

$$w_p(P,Q)=\|q(P,\cdot)-q(Q,\cdot)\|_p.$$

Theorem 7.2. *Let $p \in [1,\infty)$. The mapping $\Psi : (\mathcal{P}_{[a,b]}, w_p) \to (\mathcal{F}_c(\mathbb{R}), d_p)$ given by*

$$\Psi(P)(x)=\begin{cases} P((-\infty,x]), & x \leq b \\ 0, & x > b \end{cases}$$

is an isometry.

Proof. Fix arbitrary $\alpha \in (0,1]$ and $P \in \mathcal{P}_{[a,b]}$. From the definition of Ψ,

$$\Psi(P)_\alpha=\{x \in \mathbb{R} \mid P((-\infty,x]) \geq \alpha\} \cap (-\infty,b].$$

The infimum of $\Psi(P)_\alpha$ is therefore $q(P,\alpha)$. Since the cumulative distribution function is right continuous, whenever $x_n \to q(P,\alpha)^+$

$$P((-\infty,q(P,\alpha)])=\lim_n P((-\infty,x_n]) \geq \alpha$$

so the infimum is attained. Thus

$$\Psi(P)_\alpha=[q(P,\alpha),\infty) \cap (-\infty,b]=[q(P,\alpha),b].$$

Since the α-cuts of $\Psi(P)$ are non-empty compact intervals, to show $\Psi(P) \in \mathcal{F}_c(\mathbb{R})$ we just need to check that $\Psi(P)_0$ is bounded. But

$$\Psi(P)_0=\mathrm{cl}\bigcup_{\alpha\in(0,1]}[q(P,\alpha),b]=\mathrm{cl}(\inf_{\alpha\in(0,1]} q(P,\alpha),b] \subseteq [a,b].$$

Therefore Ψ is well defined. To prove it is an isometry, take $P, Q \in \mathcal{P}_{[a,b]}$. Then

$$d_p(\Psi(P), \Psi(Q)) = (\int_0^1 d_H([q(P,\alpha), b], [q(Q,\alpha), b])^p d\alpha)^{1/p}$$

$$= (\int_0^1 \max\{|q(P,\alpha) - q(Q,\alpha)|, |b-b|\}^p d\alpha)^{1/p} = \|q(P,\cdot) - q(Q,\cdot)\|_p = w_p(P,Q)$$

by Lemma 7.1. □

Proposition 7.3. *For any $P, Q \in \mathcal{P}_{[a,b]}$,*

$$\Psi(\mathrm{Vin}(P,Q)) = 2^{-1}(\Psi(P) + \Psi(Q))$$

where the addition and product by a scalar are the ordinary operations in $\mathcal{F}_c(\mathbb{R})$.

Proof. Let $\alpha \in (0, 1]$. From the proof of Theorem 7.2,

$$\Psi(\mathrm{Vin}(P,Q))_\alpha = [q(\mathrm{Vin}(P,Q),\alpha), b] = [2^{-1}(q(P,\alpha) + q(Q,\alpha)), b]$$
$$= 2^{-1}([q(P,\alpha), b] + [q(Q,\alpha), b]) = 2^{-1}(\Psi(P)_\alpha + \Psi(Q)_\alpha) = (2^{-1}(\Psi(P) + \Psi(Q)))_\alpha. \quad \square$$

Then we have the following consistency result for generalized vincentiles.

Theorem 7.4. *Let $P \in \mathcal{P}_{[a,b]}$ and let $\widehat{P}_n$ and $\widehat{Q}_n$ be weakly consistent estimators of P in w_p. Then, for all $p \in [1,\infty)$, the vincentized estimator is weakly w_p-consistent. In particular, the generalized vincentile function is a weakly consistent estimator of the quantile function of P in the L^p-norm.*

Proof. Notice $\Psi(\widehat{P}_n)$ and $\Psi(\widehat{Q}_n)$ are fuzzy random variables. That is so because $\widehat{P}_n$ is a random element of the compact metric space $(\mathcal{P}_{[a,b]}, w_p)$. Since $\Psi(\mathcal{P}_{[a,b]})$ is then d_p-compact, it is Borel measurable in $\mathcal{F}_c(\mathbb{R})$. Thus $\Psi(\widehat{P}_n)$ is measurable with respect to the Borel σ-algebra of d_p, i.e., it is a fuzzy random variable (Proposition 2.2). By the assumption of weak consistency and the isometry,

$$\Psi(\widehat{P}_n) \to \Psi(P), \quad \Psi(\widehat{Q}_n) \to \Psi(P)$$

in distribution in d_p. By parts (a) and (c) of Corollary 5.7,

$$2^{-1}(\Psi(\widehat{P}_n) + \Psi(\widehat{Q}_n)) \to 2^{-1}(\Psi(P) + \Psi(P)) = \Psi(P)$$

in distribution in d_p, or equivalently in probability since the limit is a non-random element of $\mathcal{F}_c(\mathbb{R})$.

By Theorem 7.2 and Proposition 7.3,

$$w_p(\mathrm{Vin}(\widehat{P}_n, \widehat{Q}_n), P) = d_p(\Psi(\mathrm{Vin}(\widehat{P}_n, \widehat{Q}_n)), \Psi(P))$$
$$= d_p(2^{-1}(\Psi(\widehat{P}_n) + \Psi(\widehat{Q}_n)), \Psi(P)) \to 0$$

in probability.

The second part follows from Lemma 7.1. □

Remark 7.1. If $\widehat{P}_n$ and $\widehat{Q}_n$ are the empirical distributions generated from two i.i.d. samples from a common distribution P, then they are weakly w_p-consistent estimators of P (even strongly consistent, see [18, Proposition 2.2.6].)

Vincentization is based on splitting a sample of size $2n$ (or generally kn) into subsamples of equal size and averaging their quantiles. From Corollary 5.7, Theorem 7.4 remains true if this procedure is replaced by randomly deciding the subsample to which each element of the sample is assigned, and weighing accordingly each estimator in the definition of the generalized vincentile.

Remark 7.2. This section aims at illustrating both the Slutski theorem for fuzzy random variables and the fact that a random probability distribution in $\mathbb{R}$ can be identified with a fuzzy random variable via the mapping Ψ. Moreover, ordinary averages of fuzzy sets correspond to averaging quantiles and match the statistical concept of vincentizing distributions.

In this connection, it can also be noticed that (in a different language) quantile averaging corresponds to the Fréchet mean [8] with respect to the w_2-metric. A typical example of the nice properties of averaging distributions using the L^2-Wasserstein metric observes that the Wasserstein average of two normals $\mathcal{N}(\mu_1, \sigma)$ and $\mathcal{N}(\mu_2, \sigma)$ is $\mathcal{N}((\mu_1 + \mu_2)/2, \sigma)$, whereas their setwise average is a (non-normal) mixture. In the end, this parallels the same reasoning that justified vincentiles (e.g., pooling unimodal data leads to losing unimodality).

The fact that adding the fuzzy sets $\Psi(P_\xi)$ and $\Psi(P_\eta)$ does not yield $\Psi(P_{\xi+\eta})$ is not a flaw of the mapping Ψ, since it yields a different operation between distributions, of the kind studied by Schweizer and Sklar [24, p. 99], which as we remark has a natural interpretation and positive features in more than one context (vincentization, Fréchet means, probabilistic metric spaces).

8. Concluding remarks

In this paper we have considered the maximum as the triangular conorm defining the union operation, and the definition of the fuzzy Cartesian product is also tied to using the minimum as the intersection. Similarly, the definitions of the arithmetics operations are consistent with Zadeh's extension principle in which suprema of minima are calculated. The results in this paper can potentially be developed in the direction of studying alternative triangular norms and conorms, other than the pair minimum/maximum.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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