

HWG for Coulomb branch of $3d$ Sicilian theory mirrors

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Abstract

Certain star shaped quivers exhibit a pattern of symmetry enhancement on the Coulomb branch of $3d$ $\mathcal{N} = 4$ supersymmetric gauge theories. This paper studies a subclass of theories where such global symmetry enhancement occurs through a computation of the Highest Weight Generating Function (HWG) and of the corresponding Hilbert Series (HS), providing a further test of the Coulomb branch formula [1]. This special subclass has a feature in which the HWG takes a particularly simple form, as a simple rational function which is either a product of simple poles (termed freely generated) or a simple PE (termed complete intersection). Out of all possible star shaped quivers, this is a particularly simple subclass. The present study motivates a further study of identifying all star shaped quivers for which their HWG is of this simple form.

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1 Introduction

Recently a general formula which allows to count BPS gauge invariant operators for the Coulomb branch of $3d \mathcal{N} = 4$ theories has been introduced [1]. This result is very remarkable since the structure of the chiral ring associated with the Coulomb branch of $3d \mathcal{N} = 4$ is quite involved. This is due to the fact that also monopole operators are present in addition to the classical fields in the Lagrangian. However, as stated above, the so-called *monopole formula* [1] allows to describe the Coulomb branch using monopole operators dressed with scalar fields from the vector multiplet. This formula reproduces the *Hilbert Series* (HS) for the Coulomb branch, i.e. the generating function which counts chiral operators present in the theory according to their dimension and other quantum numbers under global symmetries (see e.g. [2] for an introduction to this topic). This formula can be applied to any gauge theory that is good or ugly in the sense of [3]. Recently a new technique that simplifies the computation of the HS for “good” theories has been worked out [4, 5]. This novel approach relies on the notion of *Hilbert basis*, that is a sufficient set of monopole operators that generates the chiral ring and whose knowledge completely determine the HS. Moreover the application of the monopole formula led to an expression for the Coulomb branch Hilbert Series of the $T_\rho(G)$ theory in terms of the Hall-Littlewood polynomials [6] and for the Coulomb branch HS of $T_\rho^\sigma(G)$ theory in terms of the so called generalized Hall-Littlewood polynomials [7]. Moreover this formula has been successfully applied also in the context of the mirror of $3d$ Sicilian theories [8, 9]. These theories arise from the compactification of the $6d (2,0)$ theory with symmetry group G on a circle times a Riemann surface with punctures. As we will review the HS of these theories can be obtained by gluing together different $T_\rho(G)$ theories.

Moreover recently it has been developed a new mathematical tool that simplifies the computation of the HS, the so called *Highest Weight Generating function* (HWG) (see [10] for an introduction to this topic). This method is based on the highest weight Dynkin labels of the symmetry group that characterizes the theory taken under consideration and it has already been successfully applied [11][12].

In the present paper we move a further step in this direction and we perform the computation of the HWG and of the corresponding HS for the mirror of certain $3d$ Sicilian theories [8], which are chosen such that they exhibit a sufficiently large global symmetry. In particular we examine how the HS can be decomposed under representations of the global symmetry group that characterizes these theories.

The present paper is organized as follows. In section 2, after a short review of the Coulomb branch formula introduced in [1] and its application in the context of the $T_\rho(G)$ theory [6], we examine how such formula can be applied for the computation of the HS of the mirror of $3d$ Sicilian theories [7]. Moreover we also summarize the basic aspects of the computation technique that we employ in the following part of the paper. In section 3 we review the relation between the Coulomb and Higgs branch of $3d$ $\mathcal{N} = 4$ theories and closure of nilpotent orbits [13, 14]. In section 4 we summarize our main results, i.e. the general expressions of the HWG for the theories that have been taken into account. We focus our attention on theories with unitary and orthogonal global symmetry groups and on the mirror of the $(k) - [2N]$ theory (see [15, 16]). Then in section 5 we test the previous expressions performing the explicit computation of the HWG and of the Plethystic Logarithm (PLog) for theories with unitary global symmetry group. We examine in detail the cases in which the integer N , that characterizes the theory, is equal to 3 and 4. While we refer the reader to the appendix B for the analysis performed for higher values of N . Then in section 6 we test the expression of the HWG for theories with orthogonal global symmetry group. Finally we end up with some conclusions in section 7. We refer the reader to appendices A - B for the conventions that have been employed and more technical aspects related to the computations.

2 The Coulomb branch Hilbert Series for $3d$ $\mathcal{N} = 4$ theories

In all this paper we consider the Coulomb branch of $3d$ $\mathcal{N} = 4$ gauge theories. This branch is described by the VEVs of the triplet of scalar fields in the $\mathcal{N} = 4$ vector multiplet and by the VEV of the dual photons. Differently from the Higgs branch the Coulomb branch is affected by quantum corrections and the corresponding chiral ring also involves monopole operators. A suitable description of the chiral ring on the Coulomb branch has been introduced in [1]. As a matter of fact the gauge invariant objects in this branch are monopole operators dressed by a product of certain scalar fields in the vector multiplet. This provided a systematic way to study the chiral ring of this branch. Moreover an analytic expression of the corresponding generating function, known as the *Hilbert Series* (HS), has been found. This function counts gauge invariant BPS operators that have a non-zero vacuum expectation value along the Coulomb branch. In the following we denote this expression as the *monopole formula*. We review this formula in section 2.1.

Using the monopole formula an analytic expression for the HS of the so called $T_\rho(G)$ theories [3] has been introduced in [6]. This expression holds for any classical gauge group G and for any partition ρ related to the GNO dual group G^v [17]. Moreover it has been shown that the HS can be expressed as a function of the Hall-Littlewood polynomials. In the following we denote this formula as the *Hall-Littlewood formula* and we review it in section 2.2. The previous result has been generalized in the context of the $T_\rho^\sigma(G)$ theories [7].

Finally the previous computational technique has been applied in the context of the mirror of 3-dimensional Sicilian theories [7]. The computation of the HS for this class of theories can be performed gluing together the HS for different $T_\rho(G)$ theories that share the same global symmetry group. We refer to the corresponding formula as the *gluing formula*. We review it in section 2.3. In the following part of this section we summarize the basic computational tools that have been employed in the rest of this article.

2.1 The monopole formula

The monopole formula [1] allows to count all the BPS gauge invariant operators that can acquire a non-zero VEV along the Coulomb branch, according to their dimensions and other quantum numbers. Using the $\mathcal{N} = 2$ formalism the $\mathcal{N} = 4$ vector multiplet is decomposed in a $\mathcal{N} = 2$ vector multiplet and in $\mathcal{N} = 2$ chiral multiplet transforming in the adjoint representation of the gauge group. The Hilbert Series for an ugly or a good theory with gauge group G reads [1]

$$\text{HS}_G(t, z) = \sum_{\mathbf{m} \in \Gamma_{G^v}/W_{G^v}} z^{J(\mathbf{m})} t^{2\Delta(\mathbf{m})} P_G(t; \mathbf{m}), \quad (1)$$

where the sum is taken over the magnetic charges \mathbf{m} of the monopole operator $V_{\mathbf{m}}$ that, modulo a gauge transformation, belongs to a Weyl Chamber of the weight of lattice Γ_{G^v} of the GNO dual group [17]. The factor $P_G(t, \mathbf{m})$ counts operators constructed by the adjoint scalar field ϕ in the chiral multiplet. These operators are gauge invariant under the action of the gauge group $H_{\mathbf{m}}$ unbroken in the presence of the monopole operator $V_{\mathbf{m}}$. This factor is given by

$$P_G(t; \mathbf{m}) = \prod_{i=1}^r \frac{1}{1 - t^{2d_i(\mathbf{m})}}, \quad (2)$$

where the $d_i(\mathbf{m})$ are the degrees of the independent Casimir invariants of $H_{\mathbf{m}}$. Finally $\Delta(\mathbf{m})$ is the dimension of the monopole operator

$$\Delta(\mathbf{m}) = - \sum_{\alpha \in \Delta_+(G)} |\alpha(\mathbf{m})| + \frac{1}{2} \sum_{i=1}^n \sum_{\rho_i \in R_i} |\rho_i(\mathbf{m})|, \quad (3)$$

where α are the positive roots of the gauge group G and $\rho_i \in R_i$ are the weights of the matter field representation R_i under the gauge group. $J(\mathbf{m})$ is the topological charge, one per each gauge node in the quiver, of the monopole operator of GNO charges \mathbf{m} . Finally z is the fugacity of the topological symmetry.

The formula (1) can be generalized to also include background monopole fluxes for a global flavour symmetry G_F acting on the matter fields. The corresponding Hilbert Series formula reads [6]

$$\text{HS}_{G, G_F}(t, \mathbf{m}_F, z) = \sum_{\mathbf{m} \in \Gamma_{G^v}/W_{G^v}} t^{2\Delta(\mathbf{m}, \mathbf{m}_F)} P_G(t; \mathbf{m}) z^{J(\mathbf{m})}, \quad (4)$$

where the sum is taken only over the magnetic fluxes of the gauge group G but depends on the weights \mathbf{m}_F of the dual flavor group G_F^v . These weights enter in the formula (4) through the dimension Δ of the operators. Moreover, using the global symmetry, we can restrict the possible values of \mathbf{m}_F to a Weyl chamber of G_F^v and take $\mathbf{m}_F \in \Gamma_{G_F^v}/W_{G_F^v}$.

2.2 The Hall-Littlewood formula

The Hilbert Series formula (4) can be applied in the context of the $T_\rho(G)$ theory, leading to the Hall-Littlewood formula [6]. A $T_\rho(G)$ is specified by a partition ρ and classical gauge group G .¹ The partition ρ of N is given by

$$\rho = (N - N_1, N_1 - N_2, \dots, N_{d-1} - N_d, N_d), \quad (5)$$

moreover the corresponding theory is “good” (in the sense of [3]) if the partitions satisfies the non-increasing constraint

$$N - N_1 \geq N_1 - N_2 \geq N_2 - N_3 \geq \dots \geq N_{d-1} - N_d \geq N_d > 0. \quad (6)$$

The quiver diagram for this theory is reported in figure 1. The quiver theory can be obtained

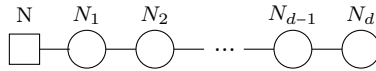


Figure 1: Quiver diagram of the $T_\rho(SU(N))$ theory.

starting from a brane configurations as proposed in [16]. The Hilbert Series for the Coulomb branch of this theory can be expressed in terms of the Hall-Littlewood polynomials as [6]

$$\text{HS}[T_\rho(SU(N))](t; x_1, \dots, x_{d+1}, n_1, \dots, n_N) = t^{\delta(\mathbf{n})} (1 - t^2)^N K_\rho(\mathbf{x}; t) \Psi_{U(N)}^{\mathbf{n}}(\mathbf{x}t^{\mathbf{w}_\rho}; t), \quad (7)$$

¹For the purpose of this paper the only relevant case is $G = SU(N)$. Therefore henceforth we focus only on this specific case.

where n_1, n_2, \dots, n_N are the background GNO charges for the $U(N)$ group, with

$$n_1 \geq n_2 \geq \dots \geq n_{N-1} \geq n_N \geq 0, \quad (8)$$

and the Hall-Littlewood polynomials are given by

$$\Psi_{U(N)}^{(n_1, \dots, n_N)}(x_1, \dots, x_N; t) = \sum_{\sigma \in S_N} x_{\sigma(1)}^{n_1} \dots x_{\sigma(N)}^{n_N} \prod_{1 \leq i < j \leq N} \frac{1 - t^2 x_{\sigma(i)}^{-1} x_{\sigma(j)}}{1 - x_{\sigma(i)}^{-1} x_{\sigma(j)}}, \quad (9)$$

while the factor $\delta(\mathbf{n})$ reads

$$\delta(\mathbf{n}) = \sum_{j=1}^N (N+1-2j)n_j. \quad (10)$$

The function $K_\rho(\mathbf{x}; t)$ depends on the particular partition ρ that have been considered. Explicitly it reads

$$K_\rho(\mathbf{x}; t) = \prod_{i=1}^{\text{length}(\rho^T)} \prod_{j,k=1}^{\rho_i^T} \frac{1}{1 - a_j^i \bar{a}_k^i}, \quad (11)$$

where ρ^T is the transpose of the partition ρ and the two factors a_j^i and \bar{a}_k^i are given by

$$\begin{aligned} a_j^i &= x_j t^{\rho_j - i + 1}, & i &= 1, \dots, \rho_j, \\ \bar{a}_k^i &= x_k^{-1} t^{\rho_k - i + 1}, & i &= 1, \dots, \rho_k, \end{aligned} \quad (12)$$

these factors are associated to each box in the Young tableau. Finally \mathbf{w}_r denotes the weights of the $SU(2)$ representation of dimension r

$$\mathbf{w}_r = (r-1, r-3, \dots, 3-r, 1-r), \quad (13)$$

therefore the notation $t^{\mathbf{w}_r}$ stands for the vector

$$t^{\mathbf{w}_r} = (t^{r-1}, t^{r-3}, \dots, t^{3-r}, t^{1-r}). \quad (14)$$

The formula (7) admits a generalization for other classical gauge groups. We refer the interested reader to [6] for a discussion of these cases.

2.3 Mirrors of 3d Sicilian theories and the gluing formula

In this section we review the formula for the Hilbert Series of Coulomb branch of the mirrors of 3d Sicilian theories [7]. These theories are obtained starting from the 6d (2,0) theory with symmetry group G performing a compactification over a punctured Riemann surface times a circle. Therefore these theories can be understood as the mirrors of the theories on M5-branes wrapping a circle times a punctured sphere ρ_1, \dots, ρ_n [8, 18]. These theories are described by a *star-shaped* quiver gauge theory. This is a quiver diagram with n -arms all connected through a central node. An example of star-shaped quiver with three arms is reported in figure 2. Each arm i of the quiver diagram is associated to a different $T_{\rho_i}(G)$ theory.

A formula that allows to obtain the Coulomb branch Hilbert Series for these class of theories have been introduced in [7]. It was demonstrated that the Hilbert Series for the Coulomb branch of the mirrors of the 3d Sicilian theories is obtained by “gluing” together the Hilbert Series for the different $T_\rho(G)$ theories associated to the different arms of the quiver diagram and that share the same global symmetry group. Specifically given a set of theories $\{T_{\rho_1}(G), \dots, T_{\rho_e}(G)\}$ all with the same global symmetry group G we can construct the corresponding mirror theory gauging the common centerless flavor symmetry $G/Z(G)$. Henceforth we restrict our attention to the particular case in which $G = SU(N)$ and we refer the reader to [7] for the discussion of the general case. In this particular case the Hilbert Series of the resulting theory reads

$$\begin{aligned} \text{HS}(t; \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(e)}) &= \\ &= \sum_{n_1 \geq \dots \geq n_N \geq 0} \prod_{j=1}^e \text{HS}[T_{\rho_j}(SU(N))](t; \mathbf{x}^{(j)}; n_1, \dots, n_N) t^{\delta(\mathbf{n})} (1-t^2) P_{U(N)}(t; n_1, \dots, n_N), \end{aligned} \quad (15)$$

we observe that the Hilbert Series of the resulting theory is obtained multiplying the Hilbert Series of the building blocks of the star-shaped quiver (i.e. the different $T_{\rho_i}(G)$ theories) and then summing over the monopoles of the gauged $SU(N)$ group. See figure 2 for a graphical representation of this formula. In this paper we consider only star-shaped quiver with three arms.

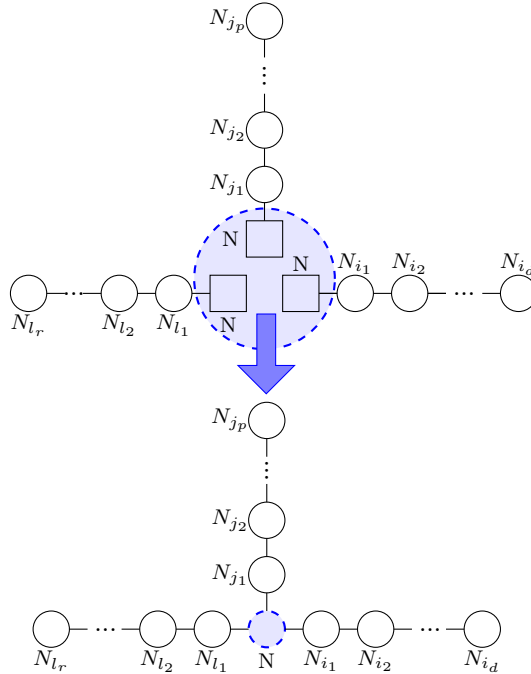


Figure 2: Graphical representation of the *gluing technique*, in the particular case in which only three $T_{\rho_i}[SU(N)]$ theories are involved.

Remarkably the global symmetry group G_{global} of these theories can be extracted directly from their quiver diagram using the following procedure [3, 19]:

1. Identify all the so called *balanced-nodes* of the quiver (i.e. a node for which the sum over the ranks k_i of the adjacent nodes is equal to $2k$, where k is the rank of the node taken in consideration).
2. If all nodes are gauged ungauged a $U(1)$ by choice.
3. Then the balanced nodes will form the Dynkin diagram of the semi-simple part of G_{global} . The abelian part of the global symmetry group is $U(1)^{k-1}$, where k is the number of unbalanced nodes of the quiver.

In the following we apply the above prescription to the five families of quiver gauge theories reported in section 4.

3 Coulomb branch and nilpotent orbits

Following [20] in this section we summarize the basic information regarding nilpotent orbits that will be relevant in the following parts of this paper. Recently it has been understood that the Coulomb branch and the Higgs branch of a $3d \mathcal{N} = 4$ theory can be related to nilpotent orbits [20]. As a matter of fact Namikawa's theorem [14] states that if the Coulomb branch or the Higgs branch is finitely generated by operators with spin one under the $SU(2)_R$ symmetry group then this space is the closure of a nilpotent orbit of the isometry group of the algebra. Spaces with

generators with spin higher than one can be thought as extensions of closures of nilpotent orbits. Moreover the generators with spin 1 transform in the adjoint representation of an isometry group of the variety [21].

Let's focus on nilpotent orbits of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$. These are in a one to one correspondence with the partitions of n , this is a n -tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ that satisfies

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n, \quad \text{and} \quad \sum_{i=1}^k \lambda_i = n. \quad (16)$$

An *elementary Jordan block* of order m is a $m \times m$ matrix,

$$J^m = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad (17)$$

given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ we can introduce the matrix X^λ associated to the partition λ

$$X^\lambda = \oplus_i J^{\lambda_i}. \quad (18)$$

The *nilpotent orbit* \mathcal{O}_λ corresponding to the partition λ is obtained as

$$\mathcal{O}_\lambda = G_{adj} \cdot X^\lambda, \quad (19)$$

where G_{adj} denotes the action of the adjoint group. We refer the reader to [13] for more details regarding nilpotent orbits.

In the following sections, in order to outline the relation between closures of nilpotent orbits and Coulomb branch, we consider the PLog expansion of the Hilbert Series and we set equal to zero all the generators with spin higher than one. For each theory we find nilpotent generators (with spin equal to one) that can be represented by a $N \times N$ nilpotent matrix M , such that $M^2 = 0$, and satisfying the *Jordan condition*

$$\text{Tr}[M^p] = 0 \quad \forall p \in \mathbb{N} \Leftrightarrow \text{all eigenvalues of } M \text{ are } 0. \quad (20)$$

This implies that the nilpotent matrix M can be only decomposed in the elementary Jordan blocks J^1 and J^2 . Therefore the rank of the matrix M must satisfy the constraint

$$\text{rank}[M] \leq \frac{N}{2}. \quad (21)$$

The above information allow to relate M to the closure of a nilpotent orbit of \mathfrak{sl}_n . This also implies that the nilpotent orbit which results by setting the extra generators to zero is at most of type $(2^k, 1^{N-2k})^2$ with $k < N/2$ and $k \in \mathbb{N}$.

4 Overview of the results

In this section we report our main results, i.e. the general expression of the HWG for the five families of quiver gauge theories that have been considered. We focused our attention on three families of quiver gauge theories with unitary global symmetry group (reported in section 4.1) and two families with orthogonal global symmetry group (reported in section 4.2). We report the quiver diagram and the corresponding global symmetry group of each theory.³

4.1 Theories with unitary global symmetry

4.1.1 Theories with $G_{global} = SU(2) \times SU(2N)$

These theories are described by the quiver diagram reported in figure 3

²We use the shorthand notation $(p^k) = (\underbrace{p, \dots, p}_k)$.

³Henceforth the balanced nodes of all the quiver diagrams are marked in red.

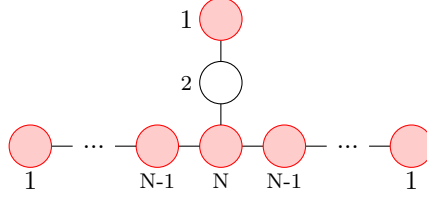


Figure 3: Quiver diagram with $SU(2N) \times SU(2)$ global symmetry group.

Note that the case $N = 3$ is special since the global symmetry is enhanced to E_6 . The HWG for this case was already discussed in [11] (see the first line of table 10 of [11]). In general the HWG for this class of theories reads

$$\text{HWG}_{SU(2) \times SU(2N)}(t; \nu, \mu_i) = \text{PE} \left[\nu^2 t^2 + t^4 + \nu \mu_N t^{N-1} + \nu \mu_N t^{N+1} + \sum_{i=1}^N \mu_i \mu_{2N-i} t^{2i} - \nu^2 \mu_N^2 t^{2N+2} \right], \quad (22)$$

where ν is the highest weight of the $SU(2)$ subgroup and the various μ_j denote the highest weights of the $SU(2N)$ subgroup.

4.1.2 Theories with $G_{\text{global}} = SU(2N)$

These theories are described by the quiver diagram reported in figure 4.

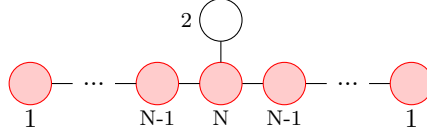


Figure 4: Quiver diagram with $SU(2N)$ global symmetry group.

Note that the case $N = 4$ is special since the global symmetry is enhanced to E_7 . The HWG for this case was already discussed in [11] (see the fourth line of table 10 of [11]). In general the HWG for this class of theories reads

$$\text{HWG}_{SU(2N)}(t; \mu_i) = \text{PE} \left[t^4 + \sum_{i=1}^{N-1} \mu_i \mu_{2N-i} t^{2i} + \mu_N t^{N-2} + \mu_N t^N \right], \quad (23)$$

where μ_j are the highest weights of the $SU(2N)$ group.

4.1.3 The mirror of the $(k) - [2N]$ theory, $G_{\text{global}} = SU(2N)$

The mirror of the $(k) - [2N]$ theory is described by the quiver reported in figure 5

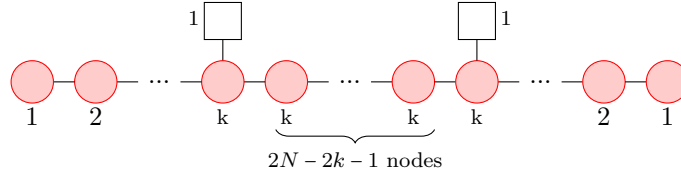


Figure 5: Quiver diagram of the mirror of the $(k) - [2N]$ theory, with $k < N$.

The corresponding HWG reads

$$\text{HWG}_{N,k}(t; \mu_i) = \text{PE} \left[\sum_{i=1}^k \mu_i \mu_{2N-i} t^{2i} \right], \quad (24)$$

where μ_j are the highest weights of the $SU(2N)$ group. The expression (24) for the HWG was already found in [22]. Moreover the relation between the Coulomb branch of this class of theories and closure of nilpotent orbits was analysed [20].

We test the expressions of the HWG (22), (23) and (24) in section 5 considering different values of N .

4.2 Theory with orthogonal global symmetry

4.2.1 Theories with $G_{Global} = SO(4N + 6) \times U(1)$

These theories are described by the quiver diagram reported in figure 6

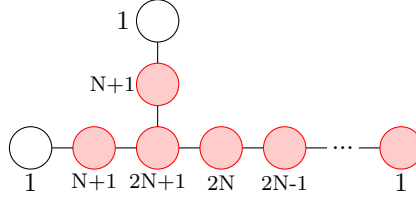


Figure 6: Quiver diagram with $SO(4N+6) \times U(1)$ global symmetry group. Note that when $N = 1$ the global symmetry is enhanced to E_6 .

We conjecture the following HWG for this class of theories

$$\text{HWG}_{SO(4N+6) \times U(1)}(t; \mu_i, q) = \text{PE} \left[t^2 + \sum_{i=1}^N \mu_{2i} t^{2i} + \frac{\mu_{2N+3} t^{N+1}}{q} + q \mu_{2N+2} t^{N+1} \right], \quad (25)$$

where q is the highest weight of the $U(1)$ subgroup while the various μ_i are the highest weights of the $SO(4N + 6)$ group.

4.2.2 Theories with $G_{global} = SO(4N + 4)$, with $N \geq 3$

These theories are described by the quiver diagram reported in figure 7 ⁴

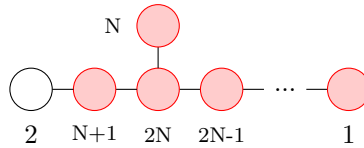


Figure 7: quiver diagram with $SO(4N + 4)$ global symmetry group.

Note that the case $N = 3$ is special since the global symmetry is enhanced to E_8 . This case was already discussed in [11] (see the sixth line of table 10 of [11]). We conjecture the following HWG for this class of theories

$$\text{HWG}_{SO(4N+4)}(t; \mu_i) = \text{PE} \left[\sum_{i=1}^N \mu_{2i} t^{2i} + t^4 + \mu_{2N+2} (t^{N-1} + t^{N+1}) \right], \quad (26)$$

where μ_i are the highest weights of $SO(4N + 4)$.

We test the expression (25) and the expression (26) in section 6.

⁴We must require $N \geq 3$ in order to ensure that all the partitions are non-increasing.

5 Theories with unitary global symmetry group

In this section we test the expressions of the HWG (22) and (23) for several values of the integer N characterizing the quiver gauge theory. For each theory we report the expression of the corresponding HWG, the first orders of the expansion of the HS and of the Pletystic logarithm, the first generators and their relations. Finally we analyse the connection between the Coulomb branch and closure of nilpotent orbits. We refer the reader to appendix B for further details regarding these computations.

5.1 $N = 3$, $G_{global} = E_6 \supset SU(2) \times SU(6)$

The quiver gauge theory with E_6 global symmetry group is reported in figure 8. We perform the

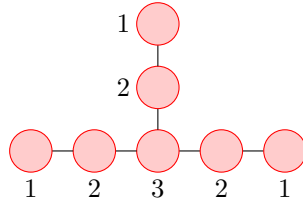


Figure 8: Quiver diagram with E_6 global symmetry group.

computation of corresponding unrefined HS using the formula (15). At the lowest orders in the t expansion we find ⁵

$$\text{HS}_{E_6}(t, 1, \dots, 1) = 1 + 78t^2 + 2430t^4 + 43758t^6 + 537966t^8 + o(t^9). \quad (27)$$

At every order in the t -expansion we decompose the E_6 representations under representation of the global symmetry subgroup $SU(2) \times SU(6)$. This way we find the HWG ⁶

$$\text{HWG}_{SU(2) \times SU(6)}(t; \nu, \mu_i) = \text{PE}[(\nu^2 + \mu_1\mu_5 + \nu\mu_3)t^2 + (1 + \mu_2\mu_4 + \mu_3\nu)t^4 + \mu_3^2t^6 - \nu^2\mu_3^2t^8]. \quad (28)$$

This result for the HWG follows the pattern outlined in equation (22). Therefore using the above HWG the corresponding HS can be written in terms of $SU(2)$ and $SU(6)$ representations as ⁷

$$\begin{aligned} \text{HS}_{SU(2) \times SU(6)}(t; x, y_i) = & 1 + ([0; 1, 0, 0, 0, 1] + [1; 0, 0, 1, 0, 0] + [2; 0, 0, 0, 0, 0])t^2 + ([0; 0, 0, 0, 0, 0] \\ & + [0; 0, 1, 0, 1, 0] + [0; 2, 0, 0, 0, 2] + [1; 0, 0, 1, 0, 0] + [1; 1, 0, 1, 0, 1] + [2; 1, 0, 0, 0, 1] + [2; 0, 0, 2, 0, 0] \\ & + [3; 0, 0, 1, 0, 0] + [4; 0, 0, 0, 0, 0])t^4 + o(t^4). \end{aligned} \quad (29)$$

The Plethystic logarithm of the Hilbert series (29) reads

$$\begin{aligned} \text{PLog}[\text{HS}_{SU(2) \times SU(6)}(t; x, y_i)] = & ([0; 1, 0, 0, 0, 1] + [1; 0, 0, 1, 0, 0] + [2; 0, 0, 0, 0, 0])t^2 + \\ & - (2[0; 0, 0, 0, 0, 0] + [0; 1, 0, 0, 0, 1] + [0; 0, 1, 0, 1, 0] + [1; 1, 1, 0, 0, 0] + [1; 0, 0, 0, 1, 1] \\ & + [1; 0, 0, 1, 0, 0] + [2; 1, 0, 0, 0, 1])t^4 + o(t^4). \end{aligned} \quad (30)$$

⁵Note that this is the Hilbert Series of the reduced moduli space of one E_6 instanton on \mathbb{C}^2 , whose computation was also performed in a different way in [23].

⁶Note that this result is in agreement with the HWG previously found in [11, 23].

⁷Henceforth the notation $[a; b_1, b_2, \dots, b_{2N-1}]$ denotes the product between the $[a]$ $SU(2)$ representation and the $[b_1, b_2, \dots, b_{2N-1}]$ $SU(2N)$ representation.

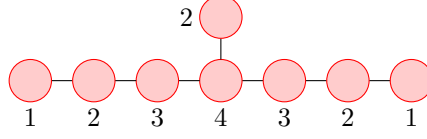


Figure 9: Quiver diagram with E_7 global symmetry group.

5.1.1 The generators and their relations

At the order t^2 of the expansion (30) we have three generators ⁸

$$[0; 1, 0, 0, 0, 1] : M_{i_2}^{i_1} \text{ with } \text{Tr}[M] = 0, \quad (31)$$

$$[1; 0, 0, 1, 0, 0] : N_{\alpha}^{[i_1 i_2 i_3]}, \quad (32)$$

$$[2; 0, 0, 0, 0, 0] : C_{\alpha\beta} \text{ with } \text{Tr}[C] = 0, \quad (33)$$

where $i_1, \dots, i_6 = 1, \dots, 6$ are $SU(6)$ indices while $\alpha, \beta = 1, 2$ are $SU(2)$ indices. The generator M transforms under the adjoint representation of $SU(6)$, the generator C transforms under the adjoint representation of $SU(2)$, while the generator $N_{\alpha}^{[i_1 i_2 i_3]}$ transforms under the completely antisymmetric representation of $SU(6)$ and under the fundamental representation of $SU(2)$.

At the order t^4 there are the following relations ⁹

$$[2; 1, 0, 0, 0, 1] : M_{j_1}^{i_1} C_{\alpha\beta} + \frac{1}{4} \left(N_{\alpha}^{[i_1 i_2 i_3]} N_{[j_1 i_2 i_3] \beta} \right) = 0, \quad (35)$$

$$[1; 1, 1, 0, 0, 0] : M_{\{i_1}^{j_1} N_{j_2 j_3\} j_1 \alpha} = 0, \quad (36)$$

$$[1; 0, 0, 0, 1, 1] : M_{j_1}^{\{i_1} N_{\alpha}^{j_2 j_3\} j_1} = 0, \quad (37)$$

$$[1; 0, 0, 1, 0, 0] : N_{\alpha}^{i_1 i_2 i_3} C_{\beta\gamma} \epsilon^{\alpha\beta} + M_{j_1}^{[i_1} N_{\gamma}^{i_2 i_3] j_1} = 0, \quad (38)$$

$$[0; 0, 1, 0, 1, 0] : (N_{\alpha}^{i_1 i_2 q} N_{j_1 j_2 q \beta} \epsilon^{\alpha\beta} - 4M_{[j_1}^{i_1} M_{j_2]}^{i_1}) |_{[0,1,0,1,0]} = 0, \quad (39)$$

$$[0; 1, 0, 0, 0, 1] : M_{j_1}^{i_1} M_{i_2}^{j_1} - \frac{1}{6} \delta_{i_2}^{i_1} M_{i_2}^{j_2} M_{j_2}^{i_2} = 0, \quad (40)$$

$$[0; 0, 0, 0, 0, 0] : N_{\alpha}^{i_1 i_2 i_3} N_{i_1 i_2 i_3 \beta} \epsilon^{\alpha\beta} + 24C_{\alpha\beta} C_{\gamma\delta} \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} = 0, \quad (41)$$

$$[0; 0, 0, 0, 0, 0] : M_{j_1}^{i_1} M_{i_1}^{j_1} + 3C_{\alpha\beta} C_{\gamma\delta} \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} = 0. \quad (42)$$

There are three generators with spin 1. The relation (40) is satisfied if M is nilpotent and if it satisfies the condition (20). The relations (41)-(42) are satisfied if C is nilpotent and if $N_{\alpha}^{ijk} N_{ijk\beta} \epsilon^{\alpha\beta} = 0$. The reduced moduli space of 1-instanton of E_6 is identified with the closure of the minimal nilpotent orbit of E_6 [25]. The previous analysis suggests that this space can be decomposed in submanifolds. As a matter of fact the nilpotent operator M is related to the closure of the minimal nilpotent orbit of $SU(6)$, i.e. to the reduced moduli space of 1-instanton of $SU(6)$. On the other hand the operator C is related to the closure of the minimal nilpotent orbit of $SU(2)$, i.e. to the reduced moduli space of 1-instanton of $SU(2)$.

5.2 $N = 4$, $G_{global} = E_7 \supset SU(8)$

The quiver gauge theory with E_7 global symmetry group is reported in figure 9. The lowest orders

⁸A similar analysis has also been carried out in [24].

⁹Where for a tensor with the structure $P_{i[jk]}$ we introduced the curly brackets $\{ \}$

$$P_{\{i[jk]\}} = P_{i[jk]} - P_{[i]jk}. \quad (34)$$

of the t expansion of the corresponding unrefined HS read ¹⁰

$$\text{HS}_{E_7}(t; 1, \dots, 1) = 1 + 133t^2 + 7371t^4 + 238602t^6 + o(t^8). \quad (43)$$

We decompose the representations of the global symmetry group E_7 under representations of its $SU(8)$ subgroup. This way we get the HWG ¹¹

$$\text{HWG}_{SU(8)}(t; \mu_i) = \text{PE}[(\mu_1\mu_7 + \mu_4)t^2 + (1 + \mu_4 + \mu_2\mu_6)t^4 + \mu_3\mu_5t^6]. \quad (44)$$

This result for the HWG follows the pattern outlined in equation (23). Therefore using the above HWG the corresponding HS can be written in terms of $SU(8)$ representations as

$$\begin{aligned} \text{HS}_{SU(8)}(t; y_i) = & 1 + ([1, 0, 0, 0, 0, 0, 1] + [0, 0, 0, 1, 0, 0, 0])t^2 + ([1, 0, 0, 1, 0, 0, 1] + [0, 1, 0, 0, 0, 1, 0] + \\ & [0, 0, 0, 1, 0, 0, 0] + [0, 0, 0, 0, 0, 0, 0] + [2, 0, 0, 0, 0, 0, 2] + [0, 0, 0, 2, 0, 0, 0])t^4 + o(t^4), \end{aligned} \quad (45)$$

the Plethystic logarithm of the HS (45) reads

$$\begin{aligned} \text{PLog}[\text{HS}_{SU(8)}(t; y_i)] = & ([1, 0, 0, 0, 0, 0, 1] + [0, 0, 0, 1, 0, 0, 0])t^2 - ([0, 0, 0, 0, 0, 0, 0] + [1, 0, 0, 0, 0, 0, 1] \\ & + [0, 1, 0, 0, 0, 1, 0] + [1, 0, 1, 0, 0, 0, 0] + [0, 0, 0, 0, 1, 0, 1])t^4 + o(t^4). \end{aligned} \quad (46)$$

5.2.1 The generators and their relations

At the order t^2 of the expansion of the PLog (46) there are two generators

$$[1, 0, 0, 0, 0, 0, 1]: M_{i_2}^{i_1} \quad \text{and} \quad \text{Tr}[M] = 0, \quad (47)$$

$$[0, 0, 0, 1, 0, 0, 0]: N^{[i_1 i_2 i_3 i_4]}, \quad (48)$$

where $i_1, \dots, i_4 = 1, \dots, 8$ are $SU(8)$ indices. The operator $M_{i_2}^{i_1}$ transforms under the adjoint representation of $SU(8)$, while the operator $N^{[i_1 i_2 i_3 i_4]}$ transforms under the representation **70** of $SU(8)$. At the order t^4 there are five relations¹²

$$[1, 0, 0, 0, 0, 0, 1] + [0, 0, 0, 0, 0, 0, 0]: M_{i_2}^{i_1} M_{i_3}^{i_2} + \delta_{i_3}^{i_1} (N^{j_1 j_2 j_3 j_4} N_{j_1 j_2 j_3 j_4}) = 0 \quad (50)$$

$$[0, 1, 0, 0, 0, 1, 0]: \left(M_{j_1}^{[i_1} M_{j_2}^{i_2]} + c_1 N^{i_1 i_2 i_3 i_4} N_{j_1 j_2 i_3 i_4} \right) |_{\mathbf{720}} = 0, \quad (51)$$

$$[1, 0, 1, 0, 0, 0, 0]: M_{j_1}^{i_1} N_{i_1 i_2 i_3 i_4} |_{\mathbf{378}} = M_{\{j_1}^{i_1} N_{i_2 i_3 i_4\} i_1} = 0, \quad (52)$$

$$[0, 0, 0, 0, 1, 0, 1]: M_{i_1}^{j_1} N^{i_1 i_2 i_3 i_4} |_{\mathbf{378}} = M^{\{j_1}_{i_1} N^{i_2 i_3 i_4\} i_1} = 0, \quad (53)$$

where $c_1 \in \mathfrak{R}$. The expression (45) is the power series expansion of the first orders of the Hilbert Series of the reduced moduli space of 1-instanton of E_7 . This space is equal to the minimal nilpotent orbit of E_7 [25]. However it's interesting to analyse the decomposition of E_7 representations under $SU(8)$ representations and interpret the corresponding relations in terms of $SU(8)$ nilpotent orbits. We note that the relation (50) is satisfied if M is a nilpotent matrix and if $N^{i_1 i_2 i_3 i_4} N_{i_1 i_2 i_3 i_4} = 0$. Since the operator M satisfies the condition (20) and has maximal rank equal to one we can relate it to the minimal nilpotent orbit of $SU(8)$.

5.3 $N = 4$, $G_{global} = SU(2) \times SU(8)$

The quiver gauge theory with $SU(2) \times SU(8)$ global symmetry group is reported in figure 10. The

¹⁰Note that this is equal to the HS of the reduced moduli space of one-instanton of E_7 on \mathbb{C}^2 . This HS has been already computed in [23].

¹¹This result agrees with the HWG found previously in [11].

¹²For a generic tensor with the structure $P_{i[jkl]}$ we define the projection to the irrep **378** as

$$P_{\{i[jkl]\}} \equiv P_{i[jkl]} - P_{[ijk]l}. \quad (49)$$

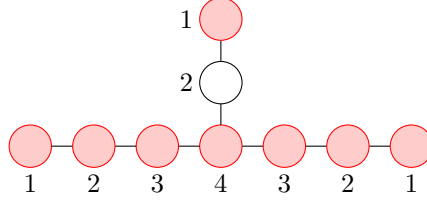


Figure 10: Quiver diagram with $SU(2) \times SU(8)$ global symmetry group.

lowest orders of the expansion of its unrefined HS are

$$\text{HS}_{SU(2) \times SU(8)}(t; 1, \dots, 1)_{SU(2) \times SU(8)} = 1 + 66t^2 + 140t^3 + 2147t^4 + 7588t^5 + 51247t^6 + o(t^6). \quad (54)$$

We decompose the previous HS under representations of $SU(2) \times SU(8)$ and we find the HWG

$$\text{HWG}_{SU(2) \times SU(8)}(t; \nu, \mu_i) = \text{PE}[\nu^2 t^2 + \mu_1 \mu_7 t^2 + \mu_4 \nu t^3 + t^4 + \mu_2 \mu_6 t^4 + \mu_4 \nu t^5 + \mu_3 \mu_5 t^6 + \mu_4^2 t^8 - \nu^2 \mu_4^2 t^{10}]. \quad (55)$$

This result for the HWG follows the pattern outlined in equation (22). The first orders of the expansion of the corresponding HS read

$$\begin{aligned} \text{HS}_{SU(2) \times SU(8)}(t; x, y_i) = & 1 + ([0; 1, 0, 0, 0, 0, 0, 1] + [2; 0, 0, 0, 0, 0, 0, 0])t^2 + [1; 0, 0, 0, 1, 0, 0, 0]t^3 + \\ & ([0; 0, 0, 0, 0, 0, 0, 0] + [0; 0, 1, 0, 0, 0, 0, 1] + [0; 2, 0, 0, 0, 0, 0, 2] + [2; 1, 0, 0, 0, 0, 0, 1] + [4; 0, 0, 0, 0, 0, 0, 0])t^4 + \\ & ([1; 1, 0, 0, 1, 0, 0, 1] + [3; 0, 0, 0, 1, 0, 0, 0] + [1; 0, 0, 0, 1, 0, 0, 0])t^5 + ([0; 0, 0, 0, 1, 0, 0, 0] + [2; 0, 0, 0, 2, 0, 0, 0]) + \\ & [0; 1, 0, 0, 0, 0, 0, 1] + [0; 1, 1, 0, 0, 0, 0, 1] + [0; 3, 0, 0, 0, 0, 0, 3] + [2; 0, 0, 0, 0, 0, 0, 0] + [2; 0, 1, 0, 0, 0, 0, 1] + \\ & [2; 2, 0, 0, 0, 0, 0, 2] + [4; 1, 0, 0, 0, 0, 0, 1] + [6; 0, 0, 0, 0, 0, 0, 0])t^6 + o(t^6). \end{aligned} \quad (56)$$

The first orders of the expansion of the PLog read

$$\begin{aligned} \text{PLog}[\text{HS}_{SU(2) \times SU(8)}(t; x, y_i)] = & ([2; 0, 0, 0, 0, 0, 0, 0] + [0; 1, 0, 0, 0, 0, 0, 1])t^2 + [1; 0, 0, 0, 1, 0, 0, 0]t^3 \\ & - ([0; 1, 0, 0, 0, 0, 0, 1] + [0; 0, 0, 0, 0, 0, 0, 0])t^4 - ([1; 0, 0, 0, 1, 0, 0, 0] + [1; 1, 0, 1, 0, 0, 0, 0] + \\ & [1; 0, 0, 0, 0, 1, 0, 1])t^5 - ([2; 0, 1, 0, 0, 0, 0, 1] + [2; 0, 0, 0, 0, 0, 0, 0] + [0; 0, 0, 1, 0, 1, 0, 0])t^6 + o(t^6). \end{aligned} \quad (57)$$

5.3.1 The generators and their relations

At the order t^2 of the expansion (57) there are two generators

$$[0; 1, 0, 0, 0, 0, 0, 1]: M_{i_2}^{i_1} \quad \text{and} \quad \text{Tr}[M] = 0, \quad (58)$$

$$[2; 0, 0, 0, 0, 0, 0, 0]: C_{\alpha\beta}, \quad (59)$$

where $i_1, i_2 = 1, \dots, 8$ are $SU(8)$ indices while $\alpha, \beta = 1, 2$ are $SU(2)$ indices. At the order t^3 there is a further generator

$$[1; 0, 0, 0, 1, 0, 0, 0]: N_{\alpha}^{[i_1 i_2 i_3 i_4]}, \quad (60)$$

this generator transforms under the $\mathbf{2} \times \mathbf{70}$ representation of $SU(2) \times SU(8)$. At the order t^4 there are the relations

$$[0; 0, 0, 0, 0, 0, 0, 0] + [0; 1, 0, 0, 0, 0, 0, 1]: M_{j_1}^{i_1} M_{i_2}^{j_1} = c_1 \delta_{i_2}^{i_1} C_{\alpha\beta} C^{\alpha\beta}, \quad (61)$$

$$(62)$$

where $c_1 \in \mathbb{R}$. At the order t^5 there are the further relations¹³

$$[1; 0, 0, 0, 1, 0, 0, 0] : N_{\alpha}^{i_1 i_2 i_3 i_4} C_{\beta\gamma} \epsilon^{\alpha\beta} + M_{j_1}^{[i_1} N_{\alpha}^{i_2 i_3 i_4] j_1} = 0, \quad (64)$$

$$[1; 0, 0, 0, 0, 1, 0, 1] : M_{j_1}^{i_1} N_{\alpha}^{j_1 j_2 j_3 j_4} |_{2 \times \mathbf{378}} = M_{j_1}^{[i_1} N_{\alpha}^{j_2 j_3 j_4] j_1} = 0, \quad (65)$$

$$[1; 1, 0, 1, 0, 0, 0, 0] : M_{i_1}^{j_1} N_{j_1 j_2 j_3 j_4 \alpha} |_{2 \times \overline{\mathbf{378}}} = M_{\{i_1}^{j_1} N_{j_2 j_3 j_4\} j_1 \alpha} = 0, \quad (66)$$

finally at the order t^6 there are the relations

$$[2; 0, 0, 0, 0, 0, 0, 0] : N_{(\alpha}^{i_1 i_2 i_3 i_4} N_{i_1 i_2 i_3 i_4 \beta)} + C_{\alpha\beta} M_{j_1}^{i_1} M_{i_1}^{j_1} = 0, \quad (67)$$

$$[2; 0, 1, 0, 0, 0, 1, 0] : \left(N_{(\alpha}^{i_1 i_2 k l} N_{j_1 j_2 k l \beta)} + c_2 C_{\alpha\beta} M_{[j_1}^{[i_1} M_{j_2]}^{i_2]} \right) |_{3 \times \mathbf{720}} = 0, \quad (68)$$

$$[0; 0, 0, 1, 0, 1, 0, 0] : \left(N_{(\alpha}^{i_1 i_2 i_3 l} N_{j_1 j_2 j_3 l \beta)} \epsilon^{\alpha\beta} + c_3 M_{[j_1}^{[i_1} M_{j_2}^{i_2} M_{j_3]}^{i_3]} \right) |_{1 \times \mathbf{2352}} = 0, \quad (69)$$

$$(70)$$

where $c_1, c_2, c_3 \in \mathbb{R}$. In order to make contact with nilpotent orbits we set equal to zero all the generators with spin higher than 1. Therefore we only consider the generator $M_{i_2}^{i_1}$ transforming under the adjoint representation of $SU(8)$ and the generator $C_{\alpha\beta}$ transforming in the adjoint of $SU(2)$. The relations (61) imply that the generators M and C are nilpotent and satisfy the condition (20).

5.4 $N = 5$, $G_{global} = SU(10)$

The quiver gauge theory with global symmetry group $SU(10)$ is reported in figure 11. The first

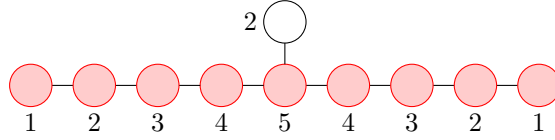


Figure 11: Quiver diagram with $SU(10)$ global symmetry group.

orders of the expansion of the corresponding unrefined Hilbert Series read

$$\text{HS}_{SU(10)}(t; 1, \dots, 1) = 1 + 99t^2 + 252t^3 + 4851t^4 + 21252t^5 + 176352t^6 + o(t^6). \quad (71)$$

We decompose the previous Hilbert Series under representation of $SU(10)$. This way we find the HWG

$$\text{HWG}_{SU(10)}(t; \mu_i) = \text{PE}[\mu_1 \mu_9 t^2 + \mu_5 t^3 + (1 + \mu_2 \mu_8) t^4 + \mu_5 t^5 + \mu_3 \mu_7 t^6 + \mu_4 \mu_6 t^8]. \quad (72)$$

This result for the HWG follows the pattern outlined in equation (23). The first orders of the expansion of the corresponding HS read

$$\begin{aligned} \text{HS}_{SU(10)}(t, y_i) = & 1 + [1, 0, 0, 0, 0, 0, 0, 0, 1] t^2 + [0, 0, 0, 0, 1, 0, 0, 0, 0] t^3 + ([0, 0, 0, 0, 0, 0, 0, 0, 0] + \\ & [0, 1, 0, 0, 0, 0, 0, 1, 0] + [2, 0, 0, 0, 0, 0, 0, 0, 2]) t^4 + ([0, 0, 0, 0, 1, 0, 0, 0, 0] + [1, 0, 0, 0, 1, 0, 0, 0, 1]) t^5 \\ & ([0, 0, 1, 0, 0, 0, 1, 0, 0] + [0, 0, 0, 0, 2, 0, 0, 0, 0] + [1, 0, 0, 0, 0, 0, 0, 0, 1] + [1, 1, 0, 0, 0, 0, 0, 0, 1, 1] \\ & [3, 0, 0, 0, 0, 0, 0, 0, 3]) t^6 + o(t^6). \end{aligned} \quad (73)$$

¹³For a generic tensor with the structure $P^{j[klm]}$ we define the projection to the irrep $\mathbf{378}$ as

$$P^{j[klm]} = P^{j[klm]} - P^{[jklm]}. \quad (63)$$

The first orders of the expansion of the PLog read

$$\begin{aligned} \text{PLog}[\text{HS}_{SU(10)}(t; y_i)] &= [1, 0, 0, 0, 0, 0, 0, 0, 1]t^2 + [0, 0, 0, 0, 1, 0, 0, 0, 0]t^3 - [1, 0, 0, 0, 0, 0, 0, 0, 1]t^4 + \\ &- ([1, 0, 0, 1, 0, 0, 0, 0, 0] + [0, 0, 0, 0, 0, 1, 0, 0, 1])t^5 - [0, 0, 1, 0, 0, 0, 1, 0, 0]t^6 + o(t^6). \end{aligned} \quad (74)$$

5.4.1 The generators and their relations

At the order t^2 of the expansion (74) there is one generator

$$[1, 0, 0, 0, 0, 0, 0, 0, 1]: \quad M^{i_1}_{i_2} \quad \text{and} \quad \text{Tr}[M] = 0, \quad (75)$$

where $i_1, i_2 = 1, \dots, 10$ are $SU(10)$ indices. This operator transforms under the adjoint representation of $SU(10)$. At the order t^3 of the expansion there is a further generator

$$[0, 0, 0, 0, 1, 0, 0, 0, 0]: \quad N^{[i_1 i_2 i_3 i_4 i_5]}, \quad (76)$$

this operators transforms under the representation **252** of $SU(10)$. At the order t^4 there is a relation

$$[1, 0, 0, 0, 0, 0, 0, 0, 1]: \quad M^{i_1}_{i_2} M^{i_2}_{i_3} - \frac{1}{10} \delta^{i_1}_{i_3} M^{i_4}_{i_5} M^{i_5}_{i_4} = 0, \quad (77)$$

this relation implies that M is a nilpotent operator. At the order t^5 there are two relations¹⁴

$$[0, 0, 0, 0, 0, 1, 0, 0, 1]: \quad M^{i_1}_{j_1} N^{j_1 i_2 i_3 i_4 i_5} |_{\mathbf{1848}} = M^{\{i_1}_{j_1} N^{i_2 i_3 i_4 i_5\} j_1} = 0, \quad (79)$$

$$[1, 0, 0, 1, 0, 0, 0, 0, 0]: \quad M^{j_1}_{i_1} N_{j_1 i_2 i_3 i_4 i_5} |_{\overline{\mathbf{1848}}} = M^{j_1}_{\{i_1} N_{i_2 i_3 i_4 i_5\} j_1} = 0, \quad (80)$$

while the relation $\overline{\mathbf{1848}}$ is given by the conjugate of the relation (79). At the order t^6 there is the relation

$$[0, 0, 1, 0, 0, 0, 1, 0, 0]: \quad (M^{[j_1}_{i_1} M^{j_2}_{i_2} M^{j_3}_{i_3}] + N^{j_1 j_2 j_3 i_4 i_5} N_{i_1 i_2 i_3 i_4 i_5}) |_{\mathbf{12375}} = 0. \quad (81)$$

In order to make contact with nilpotent orbits we set to zero the generator $N^{[i_1 i_2 i_3 i_4 i_5]}$, which has spin greater than one. Therefore M is the only nilpotent generator of spin 1.

5.5 $N = 5$, $G_{global} = SU(2) \times SU(10)$

The quiver gauge theory with $SU(2) \times SU(10)$ global symmetry group is reported in figure 12. The first orders of the expansion of the unrefined HS are

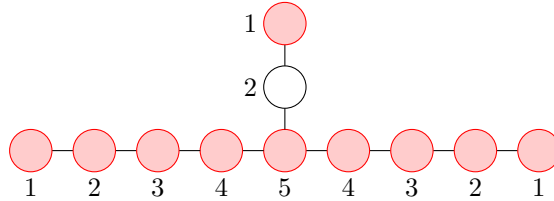


Figure 12: Quiver diagram with global symmetry group $SU(2) \times SU(10)$.

$$\text{HS}_{SU(2) \times SU(10)}(t; 1, \dots, 1) = 1 + 102t^2 + 5657t^4 + 215515t^6 + o(t^6). \quad (82)$$

¹⁴For a tensor with the structure $P^{i[jklp]}$ we define the projection to the irreducible representation **1848** as

$$P^{i[jklp]} = P^{i[jklp]} - P^{i[jklp]}. \quad (78)$$

We decompose the Hilbert Series under representation of $SU(2) \times SU(10)$. This way we find the HWG

$$\text{HWG}_{SU(2) \times SU(10)}(t; \nu, \mu_i) = \text{PE}[t^2(\mu_1\mu_9 + \nu^2) + t^4(1 + \mu_2\mu_8 + \nu\mu_5) + t^6(\nu\mu_5 + \mu_3\mu_7) + t^8\mu_4\mu_6 + t^{10}\mu_5^2 - t^{12}\nu^2\mu_5^2]. \quad (83)$$

This result for the HWG follows the pattern outlined in equation (22). The first orders of the expansion of the Hilbert Series under $SU(2) \times SU(10)$ representations read

$$\begin{aligned} \text{HS}_{SU(2) \times SU(10)}(t; x, y_i) = & 1 + ([0; 1, 0, 0, 0, 0, 0, 0, 0, 1] + [2; 0, 0, 0, 0, 0, 0, 0, 0, 0])t^2 + \\ & + ([0; 0, 0, 0, 0, 0, 0, 0, 0, 0] + [1; 0, 0, 0, 0, 1, 0, 0, 0, 0] + [0; 0, 1, 0, 0, 0, 0, 0, 0, 1, 0] + [4; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]) \\ & + [2; 1, 0, 0, 0, 0, 0, 0, 0, 1] + [0; 2, 0, 0, 0, 0, 0, 0, 0, 2]t^4 + ([0; 0, 0, 1, 0, 0, 0, 1, 0, 0] + [2; 0, 0, 0, 0, 0, 0, 0, 0, 0]) \\ & [3; 0, 0, 0, 0, 1, 0, 0, 0, 0] + [2; 0, 1, 0, 0, 0, 0, 0, 1, 0] + [6; 0, 0, 0, 0, 0, 0, 0, 0, 0] + [0; 1, 0, 0, 0, 0, 0, 0, 0, 1] \\ & + [1; 1, 0, 0, 0, 1, 0, 0, 0, 1] + [0; 1, 1, 0, 0, 0, 0, 0, 1, 1] + [4; 1, 0, 0, 0, 0, 0, 0, 0, 1] + [2; 2, 0, 0, 0, 0, 0, 0, 0, 2] \\ & [0; 3, 0, 0, 0, 0, 0, 0, 0, 3] + [1; 0, 0, 0, 0, 1, 0, 0, 0, 0]t^6 + o(t^6). \end{aligned} \quad (84)$$

The Plethystic logarithm of the Hilbert series reads

$$\begin{aligned} \text{PLog}[\text{HS}_{SU(2) \times SU(10)}(t; x, y_i)] = & ([0; 1, 0, 0, 0, 0, 0, 0, 0, 1] + [2; 0, 0, 0, 0, 0, 0, 0, 0, 0])t^2 + ([1; 0, 0, 0, 0, 1, 0, 0, 0, 0] - \\ & [0; 1, 0, 0, 0, 0, 0, 0, 0, 1] - [0; 0, 0, 0, 0, 0, 0, 0, 0, 0])t^4 + ([0; 1, 0, 0, 0, 0, 0, 0, 0, 1] - [1; 0, 0, 0, 0, 1, 0, 0, 0, 0] - \\ & [1; 1, 0, 0, 1, 0, 0, 0, 0, 0] - [1; 0, 0, 0, 0, 0, 1, 0, 0, 1])t^6 + o(t^6). \end{aligned} \quad (85)$$

5.5.1 The generators and their relations

At the order t^2 of the expansion (85) there are two generators

$$[0; 1, 0, 0, 0, 0, 0, 0, 0, 1]: M_{i_2}^{i_1} \quad \text{and} \quad \text{Tr}[M] = 0, \quad (86)$$

$$[2; 0, 0, 0, 0, 0, 0, 0, 0, 0]: C_{\alpha\beta}, \quad (87)$$

where $i_1, i_2 = 1, \dots, 10$ are $SU(10)$ indices while $\alpha, \beta = 1, 2$ are $SU(2)$ indices. The generator M transforms under the adjoint representation of $SU(10)$, while the generator C transforms under the adjoint representation of $SU(2)$. At the order t^4 there is the further generator

$$[1; 0, 0, 0, 0, 1, 0, 0, 0, 0]: N_{\alpha}^{[i_1 i_2 i_3 i_4 i_5]}, \quad (88)$$

this generator transforms under the $\mathbf{2} \times \mathbf{252}$ representation of $SU(2) \times SU(10)$. Moreover, at the same order of the expansion, there are the relations

$$[0; 0, 0, 0, 0, 0, 0, 0, 0, 0] + [0; 1, 0, 0, 0, 0, 0, 0, 0, 1]: M_{j_1}^{i_1} M_{i_2}^{j_1} = c_1 \delta_{i_2}^{i_1} C_{\alpha\beta} C^{\alpha\beta}, \quad (89)$$

where $c_1 \in \mathbb{R}$. At the order t^6 there are the following relations¹⁵

$$[1; 0, 0, 0, 0, 1, 0, 0, 0, 0]: c_2 M_{j_1}^{[i_1} N_{\gamma}^{i_2 i_3 i_4 i_5] j_1} + N_{\alpha}^{i_1 i_2 i_3 i_4 i_5} C_{\beta\gamma} \epsilon^{\alpha\beta} = 0, \quad (91)$$

$$[1; 0, 0, 0, 0, 0, 1, 0, 0, 1]: M_{j_1}^{i_1} N_{j_1 j_2 j_3 j_4 j_5}^{j_1 j_2 j_3 j_4 j_5} |_{\mathbf{2} \times \mathbf{1848}} = M_{j_1}^{[i_1} N_{\alpha}^{j_2 j_3 j_4 j_5] j_1} = 0, \quad (92)$$

$$[1; 1, 0, 0, 1, 0, 0, 0, 0, 0]: M_{i_1}^{j_1} N_{j_1 j_2 j_3 j_4 j_5} |_{\mathbf{2} \times \overline{\mathbf{1848}}} = M_{\{i_1}^{j_1} N_{j_2 j_3 j_4 j_5\} j_1 \alpha} = 0, \quad (93)$$

where $c_2 \in \mathbb{R}$. In order to make contact with nilpotent orbits we set to zero the generators with spin higher than 1. Therefore we keep only the operator $M_{i_2}^{i_1}$ transforming under the adjoint representation of $SU(10)$ and the operator $C_{\alpha\beta}$ transforming under the adjoint representation of $SU(2)$. The relations (89) implies that M and C are nilpotent operators that satisfy the condition (20).

¹⁵For a generic tensor with the structure $P_{i[jklp]}$ we define the projection to the irrep $\mathbf{1848}$ as

$$P_{\{ijklp\}} = P_{i[jklp]} - P_{[ijklp]}. \quad (90)$$

5.6 $N = 6$, $G_{global} = SU(12)$

The quiver gauge theory with $SU(12)$ global symmetry group is reported in figure 13. The first

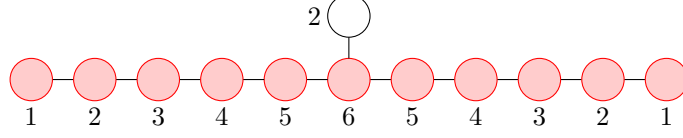


Figure 13: Quiver diagram with global symmetry group $SU(12)$.

orders of the expansion of the unrefined Hilbert Series read

$$\text{HS}_{SU(12)}(t; 1, \dots, 1) = 1 + 143t^2 + 11077t^4 + 592306t^6 + o(t^6). \quad (94)$$

We decompose the Hilbert Series under representation of $SU(12)$. This way we get the HWG

$$\text{HWG}_{SU(12)}(t; \mu_i) = \text{PE}[\mu_1 \mu_{11} t^2 + (1 + \mu_2 \mu_{10} + \mu_6) t^4 + (\mu_3 \mu_9 + \mu_6) t^6 + \mu_4 \mu_8 t^8 + \mu_5 \mu_7 t^{10}]. \quad (95)$$

This result for the HWG follows the pattern outlined in equation (23). The first orders of the expansion of the Hilbert Series in terms of $SU(12)$ are

$$\begin{aligned} \text{HS}_{SU(12)}(t; y_i) = & 1 + [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] t^2 + ([0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]) \\ & [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] + [0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0] + [2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2] t^4 + \\ & ([0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] + [0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0] + [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] \\ & + [1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1] + [1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1] + [3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3]) t^6. \end{aligned} \quad (96)$$

The first orders of the expansion of the PLog read

$$\begin{aligned} \text{PLog}[\text{HS}_{SU(12)}(t; y_i)] = & [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] t^2 + ([0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] - [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]) t^4 + \\ & ([1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] - [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] - [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1]) t^6 + o(t^6). \end{aligned} \quad (97)$$

5.6.1 The generators and their relations

At the order t^2 there is the generator

$$[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]: M_{i_2}^{i_1} \quad \text{and} \quad \text{Tr}[M] = 0, \quad (98)$$

where $i_1, i_2 = 1, \dots, 12$ are $SU(12)$ indices. This operator transforms under the adjoint representation of $SU(12)$. At the order t^4 there is another generator

$$[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0]: N^{[i_1 i_2 i_3 i_4 i_5 i_6]}, \quad (99)$$

this operator transforms under the representation **924** of $SU(12)$. Moreover there is the relation

$$[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]: M_{i_2}^{i_1} M_{i_3}^{i_2} - \frac{1}{12} \delta_{i_3}^{i_1} M_{i_3}^{i_4} M_{i_5}^{i_4} = 0, \quad (100)$$

therefore M is a nilpotent operator. At the order t^6 there are two relations¹⁶

$$[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1]: M_{j_1}^{i_1} N^{i_2 i_3 i_4 i_5 i_6 j_1} \Big|_{\mathbf{8580}} = M_{j_1}^{i_1} N^{i_2 i_3 i_4 i_5 i_6 \} j_1} = 0, \quad (102)$$

$$[1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]: M_{i_1}^{j_1} N_{i_2 i_3 i_4 i_5 i_6 j_1} \Big|_{\overline{\mathbf{8580}}} = M_{\{i_1}^{j_1} N_{i_2 i_3 i_4 i_5 i_6 \} j_1} = 0, \quad (103)$$

in order to make contact with nilpotent orbits we set to zero all the generators with spin higher than one. Therefore we keep only the nilpotent generator $M_{i_2}^{i_1}$.

¹⁶for a generic tensor with the structure $P^{i[jklpq]}$ we introduce the projection to the irreducible representation **8580** as

$$P^{\{ijklpq\}} = P^{i[jklpq]} - P^{[ijklpq]}. \quad (101)$$

5.7 $N = 6$, $G_{\text{global}} = SU(2) \times SU(12)$

The quiver gauge theory with global symmetry group $SU(2) \times SU(12)$ is reported in figure 14. The first orders of the expansion of the corresponding unrefined Hilbert Series read

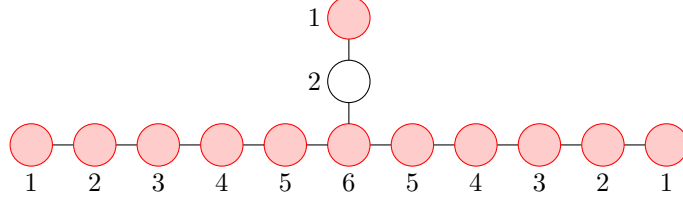


Figure 14: Quiver diagram with global symmetry group $SU(2) \times SU(12)$

$$\text{HS}_{SU(2) \times SU(12)}(t; 1, \dots, 1) = 1 + 146t^2 + 10587t^4 + 1848t^5 + 508515t^6 + o(t^6). \quad (104)$$

We decompose the Hilbert Series under representations of $SU(2) \times SU(12)$. This way we get the HWG

$$\text{HWG}_{SU(2) \times SU(12)}(t; \nu, \mu_i) = \text{PE}[(\nu^2 + \mu_1 \mu_{11})t^2 + (1 + \mu_2 \mu_{10})t^4 + \nu \mu_6 t^5 + \mu_3 \mu_9 t^6 + \nu \mu_6 t^7 + \mu_4 \mu_8 t^8 + \mu_5 \mu_7 t^{10} + \mu_6^2 t^{12} - \nu^2 \mu_6^2 t^{14}]. \quad (105)$$

This result for the HWG follows the pattern outlined in equation (22). The first orders of the expansion of the Hilbert Series in terms of $SU(2) \times SU(12)$ representations are

$$\begin{aligned} \text{HS}_{SU(2) \times SU(12)}(t; x, y_i) = & 1 + ([0; 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] + [2; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0])t^2 \\ & ([0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] + [0; 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0] + [0; 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2] + \\ & [2; 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] + [4; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0])t^4 + [1; 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0]t^5 + \\ & ([0; 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0] + [0; 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] + [0; 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1] + \\ & [0; 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3] + [2; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] + [2; 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0] \\ & [2; 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2] + [4; 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] + [6; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0])t^6 + o(t^6). \end{aligned} \quad (106)$$

The PLog of the corresponding HS reads

$$\begin{aligned} \text{PLog}[\text{HS}_{SU(2) \times SU(12)}(t; x, y_i)] = & ([0; 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] + [2; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0])t^2 + \\ & - ([0; 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] + [0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0])t^4 + [1; 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0]t^5 + \\ & [0; 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]t^6 + o(t^6). \end{aligned} \quad (107)$$

5.7.1 The generators and their relations

At the order t^2 of the expansion (107) there are two generators

$$[0; 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]: M_{i_2}^{i_1} \quad \text{and} \quad \text{Tr}[M] = 0, \quad (108)$$

$$[0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]: C_{\alpha\beta}, \quad (109)$$

where $i_1, i_2 = 1, \dots, 12$ are $SU(12)$ indices while $\alpha, \beta = 1, 2$ are $SU(2)$ indices. The generator M transforms under the adjoint representation of $SU(12)$, while the operator C transforms under the adjoint representation of $SU(2)$. At the order t^4 there are the relations

$$[0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] + [0; 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]: M_{j_1}^{i_1} M_{i_2}^{j_1} = c_1 \delta_{i_1 i_2} C_{\alpha\beta} C^{\alpha\beta}, \quad (110)$$

where $c_1 \in R$. At the order t^5 there is a further generator

$$[1; 0, 0, 0, 0, 0, 1, 0, 0, 0, 0]: N_{\alpha}^{[i_1 i_2 i_3 i_4 i_5 i_6]}, \quad (111)$$

this generator transforms under the $\mathbf{2} \times \mathbf{924}$ representation of $SU(2) \times SU(12)$.

We set equal to zero all the generators with spin higher than 1. This way we keep only the operator $M_{i_2}^{i_1}$ and the operator $C_{\alpha\beta}$. The relation (110) implies that M and C are nilpotent and satisfy the condition (20).

5.8 The mirror of the $(k) - [2N]$ theory

We test the expression (24) performing the computation of the HWG for the mirror of the $(k) - [2N]$ theory, with $k \leq N$. The relation between this class of theories and closure of nilpotent orbits has been extensively studied in [20]. The result is that each theory can be parametrized by a $N \times N$ nilpotent matrix whose rank is at most equal to k . So that the Coulomb branch \mathcal{M}_C of the mirror of the $(k) - [2N]$ can be written as the closure of the nilpotent orbit parametrized by the partition $(2^k, 1^{2N-2k})$.

$$\mathcal{M}_C = \bar{\mathcal{O}}_{(2^k, 1^{2N-2k})}. \quad (112)$$

6 Theories with orthogonal global symmetry group

In this section we test the expression of the HWG (25) when $N = 1, 2$.

6.1 $N = 1$, $G_{global} = E_6 \supset SO(10) \times U(1)$

The computation of the HS for this case is reported in section 5.1. However this time we decompose

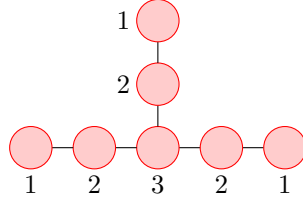


Figure 15: Quiver diagram with E_6 global symmetry group.

representations of E_6 under representations of $SO(10) \times U(1)$. This way we find the HWG ¹⁷

$$\text{HWG}_{SO(10) \times U(1)}(t; q; \mu_i) = \text{PE} \left[t^2 + \mu_2 t^2 + \mu_4 q t^2 + \frac{\mu_5}{q} t^2 \right], \quad (113)$$

where q is the fugacity for the $U(1)$ charge and the various μ_i are the $SO(10)$ highest weights. The first orders of the expansion of the corresponding Hilbert Series are

$$\begin{aligned} \text{HS}_{SO(10) \times U(1)}(t; y_i, q) = & 1 + ([0, 0, 0, 0, 0] + [0, 1, 0, 0, 0] + q^{-1}[0, 0, 0, 0, 1] + q[0, 0, 0, 1, 0])t^2 \\ & + ([0, 0, 0, 0, 0] + [0, 1, 0, 0, 0] + [0, 2, 0, 0, 0] + [0, 0, 0, 1, 1] + q^{-1}([0, 1, 0, 0, 1] + [0, 0, 0, 0, 1]) + \\ & q([0, 1, 0, 1, 0] + [0, 0, 0, 1, 0]) + q^{-2}[0, 0, 0, 0, 2] + q^2[0, 0, 0, 2, 0])t^4 + o(t^4). \end{aligned} \quad (114)$$

The first orders of the expansion of the corresponding Plethystic logarithm read

$$\begin{aligned} \text{Plog}[\text{HS}_{SO(10) \times U(1)}(t; y_i, q)] = & (1 + [0, 1, 0, 0, 0] + q^{-1}[0, 0, 0, 0, 1] + q[0, 0, 0, 1, 0])t^2 + \\ & - (2 + [0, 1, 0, 0, 0] + [2, 0, 0, 0, 0] + [0, 0, 0, 1, 1] + q([1, 0, 0, 0, 1] + [0, 0, 0, 1, 0]) + \\ & q^{-1}([1, 0, 0, 1, 0] + [0, 0, 0, 0, 1]) + (q^2 + q^{-2})[1, 0, 0, 0, 0])t^4 + o(t^4). \end{aligned} \quad (115)$$

¹⁷This HWG is in agreement with the result found in [26].

6.1.1 The generators and their relations

We use the same conventions employed in [26]. We denote $SO(10)$ vector indices with Latin letters $a, b, = 1, \dots, 10$, while we denote $SO(10)$ spinor indices with Greek letters $\alpha, \beta = 1, \dots, 16$ ¹⁸. At the order t^2 there are the following four generators

$$M^{ab}, \quad T^\alpha, \quad \tilde{T}_\alpha, \quad S, \quad (120)$$

where M^{ab} is a 10×10 antisymmetric matrix. The operator T^α transforms under the spinorial representation while \tilde{T}^α transforms under the complex conjugated representation, finally S is a scalar operator. At the order t^4 there are the following relations¹⁹

$$[2, 0, 0, 0, 0] + [0, 0, 0, 0, 0] : \quad M^{ab}M^{bc} = (T^\alpha \tilde{T}_\alpha) \delta^{ac}, \quad (121)$$

$$[0, 0, 0, 1, 1] : \quad M^{[a_1 a_2} M^{a_3 a_4]} = \tilde{T}_\beta (\gamma^{a_1 \dots a_4})^\beta_\alpha T^\alpha, \quad (122)$$

$$[0, 0, 0, 0, 0] : \quad S^2 = T^\alpha \tilde{T}_\alpha, \quad (123)$$

$$[0, 1, 0, 0, 0] : \quad SM^{ab} = \tilde{T}_\beta (\gamma^{ab})^\beta_\alpha T^\alpha, \quad (124)$$

$$q([1, 0, 0, 0, 1] + [0, 0, 0, 1, 0]) : \quad M^{ab}T^\alpha (\gamma^b)^\beta_\alpha = ST^\alpha (\gamma^a)^\beta_\alpha, \quad (125)$$

$$q^{-1}([1, 0, 0, 1, 0] + [0, 0, 0, 0, 1]) : \quad M^{ab}\tilde{T}_\beta (\gamma^b)^\beta_\alpha = S\tilde{T}_\beta (\gamma^a)^\beta_\alpha, \quad (126)$$

$$(q^2 + q^{-2})[1, 0, 0, 0, 0] : \quad T^\alpha T^\beta (\gamma^a)_{\alpha\beta} = \tilde{T}_\alpha \tilde{T}_\beta (\gamma^a)^{\alpha\beta} = 0. \quad (127)$$

6.2 $N=2$, $G_{global} = SO(14) \times U(1)$

The quiver diagram with $SO(14) \times U(1)$ global symmetry is reported in figure 16. The first orders

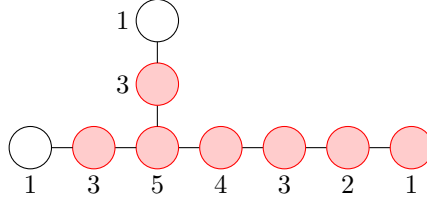


Figure 16: Quiver diagram with global symmetry group $SO(14) \times U(1)$.

of the expansion of the unrefined HS are

$$\text{HS}_{SO(14) \times U(1)}(t; 1, \dots, 1) = 1 + 92t^2 + 128t^3 + 4173t^4 + 9984t^5 + 127920t^6 + o(t^6). \quad (128)$$

The corresponding HWG reads

$$\text{HWG}_{SO(14) \times U(1)}(t; \mu_i, q) = \text{PE} \left[t^2 + \mu_2 t^2 + q \mu_6 t^3 + \frac{\mu_7}{q} t^3 + \mu_4 t^4 \right], \quad (129)$$

¹⁸The Kronecker delta has the following form

$$\delta_\beta^\alpha, \quad (116)$$

while the gamma matrices take the forms

$$(\gamma^a)_{\alpha\beta} \quad \text{and} \quad (\gamma^a)^{\alpha\beta}. \quad (117)$$

The product of two gamma matrices takes the form

$$(\gamma^{ab})_\tau^\alpha = (\gamma^{[a})^{\alpha\sigma} (\gamma^{b]})_{\sigma\tau}, \quad (118)$$

while the product of four gamma matrices reads

$$(\gamma^{abcd})_\beta^\alpha = (\gamma^{[a})^{\alpha\tau_1} (\gamma^{b])_{\tau_1\tau_2} (\gamma^{c])^{\tau_2\tau_3} (\gamma^{d]})_{\tau_3\beta}. \quad (119)$$

¹⁹A similar analysis of this moduli space has been performed in [26].

where the μ_i are $SO(14)$ highest weights while q is the $U(1)$ fugacity. The HS expressed in terms of $SO(14) \times U(1)$ representations reads

$$\begin{aligned} \text{HS}(t; y_i, q)_{SO(14) \times U(1)} = & 1 + [0, 1, 0, 0, 0, 0, 0]t^2 + (q^{-1}[0, 0, 0, 0, 0, 0, 1] + q[0, 0, 0, 0, 0, 1, 0])t^3 \\ & + (1 + [0, 0, 0, 1, 0, 0, 0] + [0, 2, 0, 0, 0, 0, 0] + [0, 1, 0, 0, 0, 0, 0])t^4 + (q^{-1}[0, 0, 0, 0, 0, 0, 1] + \\ & q^{-1}[0, 1, 0, 0, 0, 0, 1] + q[0, 0, 0, 0, 0, 1, 0] + q[0, 1, 0, 0, 0, 1, 0])t^5 + (1 + [0, 0, 0, 0, 0, 1, 1] + \\ & [0, 0, 0, 1, 0, 0, 0] + [0, 1, 0, 0, 0, 0, 0] + [0, 1, 0, 1, 0, 0, 0] + [0, 2, 0, 0, 0, 0, 0] + [0, 3, 0, 0, 0, 0, 0] \\ & + q^2[0, 0, 0, 0, 0, 2, 0] + q^{-2}[0, 0, 0, 0, 0, 0, 2])t^6 + o(t^6). \end{aligned} \quad (130)$$

The corresponding Plethystic logarithm reads

$$\begin{aligned} \text{PLog}[\text{HS}_{SO(14) \times U(1)}(t; y_i, q)] = & ([0, 1, 0, 0, 0, 0, 0] + [0, 0, 0, 0, 0, 0, 0])t^2 + \\ & + (q[0, 0, 0, 0, 0, 1, 0] + q^{-1}[0, 0, 0, 0, 0, 0, 1])t^3 - ([0, 0, 0, 0, 0, 0, 0] + [2, 0, 0, 0, 0, 0, 0])t^4 + \\ & - (q([1, 0, 0, 0, 0, 0, 1] + [0, 0, 0, 0, 0, 1, 0]) + q^{-1}([1, 0, 0, 0, 0, 1, 0] + [0, 0, 0, 0, 0, 0, 1]))t^5 + \\ & - (1 + [0, 0, 0, 0, 0, 1, 1] + [0, 0, 0, 1, 0, 0, 0] + [0, 1, 0, 0, 0, 0, 0] - [2, 0, 0, 0, 0, 0, 0] \\ & + (q^2 + q^{-2})[0, 0, 1, 0, 0, 0, 0])t^6 + o(t^6). \end{aligned} \quad (131)$$

6.2.1 The generators and their relations

At the order t^2 there are two generators

$$M^{ab} \quad \text{and} \quad S, \quad (132)$$

where M^{ab} is an antisymmetric matrix, while S is a scalar operator. At the order t^3 there two further generators

$$T^\alpha \quad \text{and} \quad \tilde{T}_\alpha, \quad (133)$$

the operator T^α transforms under the spinorial representation of $SO(14)$, while the operator \tilde{T}_α transforms under the complex conjugate representation. At the order t^4 there are the relations

$$[2, 0, 0, 0, 0, 0, 0] + [0, 0, 0, 0, 0, 0, 0] : \quad M^{ab}M^{bc} = S^2\delta^{ac}. \quad (134)$$

At the order t^5 we have the further relations

$$q([1, 0, 0, 0, 0, 0, 1] + [0, 0, 0, 0, 0, 1, 0]) : \quad M^{ab}T^\alpha(\gamma^b)^\beta_\alpha = ST^\alpha(\gamma^a)^\beta_\alpha, \quad (135)$$

$$q^{-1}([1, 0, 0, 0, 0, 1, 0] + [0, 0, 0, 0, 0, 0, 1]) : \quad M^{ab}\tilde{T}_\beta(\gamma^b)^\beta_\alpha = S\tilde{T}_\beta(\gamma^a)^\beta_\alpha. \quad (136)$$

At the order t^6 there are the relations

$$[0, 0, 0, 0, 0, 0, 0] : \quad S^3 = T^\alpha\tilde{T}_\alpha, \quad (137)$$

$$[0, 0, 0, 0, 0, 1, 1] : \quad M^{[a_1a_2}M^{a_3a_4}M^{a_5a_6]} = \tilde{T}_\beta(\gamma^{a_1\dots a_6})^\beta_\alpha T^\alpha, \quad (138)$$

$$[0, 1, 0, 0, 0, 0, 0] : \quad SM^{ab} = \tilde{T}_\beta(\gamma^{ab})^\beta_\alpha T^\alpha, \quad (139)$$

$$(q^2 + q^{-2})[0, 0, 1, 0, 0, 0, 0] : \quad T^\alpha T^\beta(\gamma^{abc})_{\alpha\beta} = \tilde{T}_\alpha\tilde{T}_\beta(\gamma^{abc})^{\alpha\beta} = 0. \quad (140)$$

6.3 $N = 3$ $G_{\text{global}} = E_8 \supset SO(16)$

The quiver gauge theory with E_8 global symmetry group is reported in figure 17.

The first orders of the expansion of the unrefined Hilbert Series are

$$\text{HS}_{SO(16)}(t) = 1 + 248t^2 + 27000t^4 + 1763125t^6 + o(t^6). \quad (141)$$

The corresponding HWG reads

$$\text{HWG}_{SO(16)}(t; \mu_i) = \text{PE}[(\mu_2 + \mu_8)t^2 + (1 + \mu_4 + \mu_8)t^4 + \mu_6t^6], \quad (142)$$

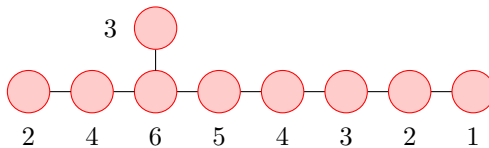


Figure 17: quiver diagram with E_8 global symmetry group.

where the μ_i are $SO(16)$ highest weights. The HS expressed in terms of $SO(16)$ representations reads

$$\begin{aligned} \text{HS}_{SO(16)}(t; \mu_i) = 1 + & ([0, 1, 0, 0, 0, 0, 0, 0] + [0, 0, 0, 0, 0, 0, 0, 1])t^2 + (1 + [0, 0, 0, 1, 0, 0, 0, 0] + \\ & [0, 0, 0, 0, 0, 0, 0, 1] + [0, 2, 0, 0, 0, 0, 0, 0] + [0, 0, 0, 0, 0, 0, 0, 2] + [0, 1, 0, 0, 0, 0, 0, 1])t^4 + o(t^4). \end{aligned} \quad (143)$$

The first orders of the expansion of the corresponding Plethystic logarithm are

$$\begin{aligned} \text{PLog}[\text{HS}_{SO(16)}(t; \mu_i)] = & ([0, 0, 0, 0, 0, 0, 0, 1] + [0, 1, 0, 0, 0, 0, 0, 0])t^2 \\ & - (1 + [0, 0, 0, 1, 0, 0, 0, 0] + [1, 0, 0, 0, 0, 0, 1, 0] + [2, 0, 0, 0, 0, 0, 0, 0])t^4 + o(t^4), \end{aligned} \quad (144)$$

6.3.1 The generators and their relations

At the order t^2 there two generators

$$X^{ab} \quad \text{and} \quad \tilde{T}_\alpha, \quad (145)$$

where we use $a, b = 1, \dots, 16$ to denote $SO(16)$ vector indices and $\alpha, \beta = 1, \dots, 128$ to denote $SO(16)$ spinor indices. The operator X^{ab} is an antisymmetric matrix in the adjoint representation of $SO(16)$. The operator \tilde{T}_α transforms under the conjugate spinor representation of $SO(16)$. At the order t^4 there are the relations

$$1 + [2, 0, 0, 0, 0, 0, 0, 0]: \quad X^2 = 0, \quad (146)$$

$$[1, 0, 0, 0, 0, 0, 1, 0]: \quad X^{ab}\tilde{T}_\beta(\gamma^b)_\alpha^\beta = 0, \quad (147)$$

$$[0, 0, 0, 1, 0, 0, 0, 0]: \quad \tilde{T}_\alpha\tilde{T}_\beta(\gamma^{abcd})^{\alpha\beta} = 0. \quad (148)$$

The relation (146) tells us that X is a nilpotent operator.

7 Conclusions

In this paper we found and we successfully tested the expressions of the HWG and of the corresponding HS for the Coulomb branch of the mirror of some families of $3d$ Sicilian theories. In particular we analysed the case of theories with unitary and orthogonal global symmetry group. In all the cases we decomposed the corresponding HWG and HS under representation of G_{global} . Moreover we explicitly checked that the numerator of the unrefined Hilbert Series is given by a palindromic polynomial and we studied the generators and the relations arising from the power series expansion of the corresponding Plethystic Logarithm. It would be interesting to extend the previous analysis also to different theories which exhibit a more involved global symmetry group. We postpone such study for future work.

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Appendices

A Notation

In all the paper we employed the following conventions

- Following [23] we denote with t the fugacity related to the R-charge of the operators.
- We denote with the Greek letters μ_i , ν_i and q the fugacities of the highest weights of the HWG. More specifically ν is related to an $SU(2)$ highest weight, while q is related to an $U(1)$ highest weight.
- We denote with x the fugacity of the $SU(2)$ global symmetry and with y_i the fugacities of the various $SU(2N)$ or $SO(N)$ global symmetry groups.

B Details of the computations

In this appendix we collect the results for higher values of the number N that parametrizes the quiver gauge theories.

B.1 $N=3$, $G_{global} = E_6 \supset SU(2) \times SU(6)$

The fundamental “building block” of the quiver diagram reported in figure 8 is the quiver [3] – (2) – (1). Using the formula (7) the corresponding Hilbert Series reads

$$\text{HS}[T_{(1,1,1)}(SU(3))](t; x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, n_1, n_2, 0) = t^{2n_1} (1-t^2)^3 \text{PE} \left[t^2 \sum_{i=1}^3 \sum_{j=1}^3 \frac{x_i}{x_j} \right] \Psi_{U(3)}^{(n_1, n_2, 0)}(x_1, x_2, x_3; t). \quad (\text{B.1})$$

We glue together three of these theories gauging the common $SU(3)$ flavour group using the formula (15). This way the HS for the quiver theory reported in figure 8 reads

$$\text{HS}_{E_6}(t, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}) = \sum_{n_1 \geq n_2 \geq n_3 = 0} t^{-2n_1} (1-t^2) P_{U(3)}(n_1, n_2, 0; t) \times \prod_{i=1}^3 \text{HS}[T_{(1,1,1)}(SU(3))](t; \mathbf{x}^{(i)}, n_1, n_2), \quad (\text{B.2})$$

where we set $n_3 = 0$ and the fugacities $\mathbf{x}^i = (x_1^i, x_2^i, x_3^i)$ satisfy the constraint

$$\prod_{k=1}^3 x_k^i = 1, \quad \text{for } i = 1, 2, 3. \quad (\text{B.3})$$

The corresponding unrefined Hilbert series reads ²⁰

$$\text{HS}_{SU(2) \times SU(6)}(t) = \frac{P_{SU(2) \times SU(6)}(t)}{(1-t^2)^{22}}, \quad (\text{B.4})$$

²⁰This computation has been performed already in [23].

where $P_{SU(2) \times SU(6)}(t)$ is a palindromic polynomial given by

$$P_{SU(2) \times SU(6)}(t) = (1+t^2)(1+55t^2+890t^4+5886t^6+17929t^8+26060t^{10}+\dots \text{(palindrome)} \dots +t^{20}). \quad (\text{B.5})$$

The dimension of the pole of the unrefined HWG (28) at $t = 1$ is 6 while the degree of the polynomial arising from the dimension of the $SU(2) \times SU(6)$ representation $[n_1+2n_2; n_3, n_4, n_1+2n_5, n_4, n_3]$ is 16. They add up to 22 which is the dimension of the reduced moduli space of one instanton of E_6 .

B.2 $N = 4$, $G_{global} = E_7 \supset SU(8)$

The fundamental ‘‘building blocks’’ of the quiver diagram reported in figure 9 are the quiver diagram [4]-(3)-(2)-(1) and the quiver diagram [4]-(2). We perform the computation of the corresponding Hilbert Series using the formula (7). For the first quiver we get

$$\begin{aligned} \text{HS}[T_{(1,1,1,1)}(SU(4))](x_1, x_2, x_3, x_4, n_1, n_2, n_3, 0) &= \\ &= t^{(3n_1+n_2-n_3)}(1-t^2)^4 \text{PE} \left[\sum_{i=1}^4 \sum_{j=1}^4 \frac{x_i}{x_j} t^2 \right] \Psi_{U(4)}^{(n_1, n_2, n_3, 0)}(x_1, x_2, x_3, x_4; t), \end{aligned} \quad (\text{B.6})$$

where the fugacities x_i satisfy the constraint $x_1 x_2 x_3 x_4 = 1$ and we set $n_4 = 0$. While for the second kind of quiver diagram we get

$$\begin{aligned} &= \text{HS}[T_{(2,2)}(SU(4))](p_1, p_2, n_1, n_2, n_3, 0) = \\ &t^{(3n_1+n_2-n_3)}(1-t^2)^4 \text{PE} \left[(2+p_1 p_2^{-1} + p_2 p_1^{-1})(t^4 + t^2) \right] \Psi_{U(4)}^{(n_1, n_2, n_3, 0)}(p_1 t, p_1 t^{-1}, p_2 t, p_2 t^{-1}; t), \end{aligned} \quad (\text{B.7})$$

where the fugacities satisfy the constraint $p_1^2 p_2^2 = 1$ and we set $n_4 = 0$. We use the formula (15) and we glue together the three quiver gauge theories gauging the common $SU(4)$ flavour group. The HS of the full quiver gauge theory reads

$$\begin{aligned} \text{HS}_{E_7}(t; x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, p_1, p_2) &= \sum_{n_1 \geq n_2 \geq n_3 \geq n_4 = 0} t^{-2(3n_1+n_2-n_3)}(1-t^2) P_{U(4)}(n_1, n_2, n_3, 0; t) \times \\ &\text{HS}[T_{(1,1,1,1)}(SU(4))](t; x_1, x_2, x_3, x_4, n_1, n_2, n_3) \times \text{HS}[T_{(1,1,1,1)}(SU(4))](t; y_1, y_2, y_3, y_4, n_1, n_2, n_3) \times \\ &\text{HS}[T_{(2,2)}(SU(4))](p_1, p_2, n_1, n_2, n_3, 0). \end{aligned} \quad (\text{B.8})$$

The unrefined Hilbert series reads²¹

$$\text{HS}_{SU(8)}(t) = \frac{P_{SU(8)}(t)}{(1-t^2)^{34}}, \quad (\text{B.9})$$

where $P_{SU(8)}(t)$ is a palindromic polynomial given by

$$\begin{aligned} P_{SU(8)}(t) &= 1 + 99t^2 + 3410t^4 + 56617t^6 + 521917t^8 + 2889898t^{10} + 10086066t^{12} + \\ &22867856t^{14} + 34289476t^{16} + \dots \text{(palindrome)} \dots + t^{34}. \end{aligned} \quad (\text{B.10})$$

The dimension of the pole of the unrefined HWG (44) at $t = 1$ is 6 while the degree of the polynomial arising from the dimension of the $SU(8)$ representation $[n_1, n_2, n_3, n_4 + n_5, n_3, n_2, n_1]$ is 28. They add up to 34 which is the dimension of the reduced moduli space of one-instanton of E_7 .

²¹This result agrees with the expression of the unrefined Hilbert Series of the moduli space of 1- E_7 instanton previously found in [23].

B.3 $N = 4$, $G_{global} = SU(2) \times SU(8)$

The fundamental “building blocks” of the quiver diagram reported in fig.10 are the quiver diagram [4]-(3)-(2)-(1) and the quiver diagram [4]-(2)-(1). We perform the computation of the corresponding HS using the formula (7). The Hilbert Series of the first quiver reads

$$\begin{aligned} \text{HS}[T_{(1,1,1,1)}(SU(4))](x_1, x_2, x_3, x_4, n_1, n_2, n_3, 0) = \\ = t^{3n_1+n_2-n_3} (1-t^2)^4 \text{PE} \left[\sum_{i=1}^4 \sum_{j=1}^4 \frac{x_i}{x_j} t^2 \right] \Psi_{U(4)}^{(n_1, n_2, n_3, 0)}(x_1, x_2, x_3, x_4; t), \end{aligned} \quad (\text{B.11})$$

where the fugacities x_i satisfy the constraint $x_1 x_2 x_3 x_4 = 1$ and we set $n_4 = 0$. While the HS of the second quiver reads

$$\begin{aligned} \text{HS}[T_{(2,1,1)}(SU(4))](p_1, p_2, p_3, n_1, n_2, n_3, 0) = \\ = t^{3n_1+n_2-n_3} (1-t^2)^4 K_{(2,1,1)}^{U(4)}(p_1, p_2, p_3, t) \Psi_{U(4)}^{(n_1, n_2, n_3, 0)}(p_1 t, p_1 t^{-1}, p_2, p_3; t), \end{aligned}$$

where the fugacities satisfy the constraint $p_1^2 p_2 p_3 = 1$ and we set $n_4 = 0$. We glue together the three quiver diagrams gauging the common $SU(4)$ global symmetry group using the formula (15). The corresponding unrefined Hilbert series reads

$$\text{HS}_{SU(2) \times SU(8)}(t, 1, \dots, 1) = \frac{P_{SU(2) \times SU(8)}(t)}{(1-t)^{36} (1+t)^{24} (1+t+t^2)^{18}}, \quad (\text{B.12})$$

where $P_{SU(2) \times SU(8)}(t)$ is a palindromic polynomial given by

$$\begin{aligned} P_{SU(2) \times SU(8)}(t) = & 1 + 6t + 63t^2 + 430t^3 + 2579t^4 + 13672t^5 + 64581t^6 + 273874t^7 + 1057876t^8 + \\ & 3739708t^9 + 12168151t^{10} + 36629984t^{11} + 102449081t^{12} + 267099092t^{13} + 651158236t^{14} + \\ & 1488399930t^{15} + 3197185885t^{16} + 6467034500t^{17} + 12340071356t^{18} + 22247726312t^{19} + \\ & 37949176435t^{20} + 61318520286t^{21} + 93953394952t^{22} + 136633528532t^{23} + 188739697078t^{24} + \\ & 247809311486t^{25} + 309426670826t^{26} + 367597436878t^{27} + 415631203373t^{28} + 447372202126t^{29} + \\ & 458475487710t^{30} + \dots + (\text{palindrome}) + \dots + t^{60}. \end{aligned} \quad (\text{B.13})$$

The dimension of the pole of the unrefined HWG (55) at $t = 1$ is 7 while the degree of the polynomial arising from the dimension of the $SU(2) \times SU(8)$ representation $[2n_1 + n_3; n_2, n_3, n_4, n_5, 2n_6 + n_3, n_5, n_4, n_2]$ is 29. They add up to 36 which is the dimension of the pole at $t = 1$ of the unrefined Hilbert Series (B.12).

B.4 $N = 5$, $G_{global} = SU(10)$

The “building blocks” of the quiver diagram reported in figure 11 are the quiver diagram [5]-(4)-(3)-(2)-(1) and the quiver diagram [5]-(2). We use the formula (7) and we compute the Hilbert Series for each of them. The Hilbert Series of the first quiver diagram reads

$$\begin{aligned} \text{HS}[T_{(1,1,1,1,1)}(SU(5))](x_1, x_2, x_3, x_4, x_5, n_1, n_2, n_3, n_4, 0) = \\ = t^{4n_1+2n_2-2n_4} (1-t^2)^5 \text{PE} \left[\sum_{i=1}^5 \sum_{j=1}^5 \frac{x_i}{x_j} t^2 \right] \Psi_{U(5)}^{(n_1, n_2, n_3, n_4, 0)}(x_1, x_2, x_3, x_4, x_5; t), \end{aligned} \quad (\text{B.14})$$

where the fugacities x_i satisfy the constraint $x_1 x_2 x_3 x_4 x_5 = 1$ and we set $n_5 = 0$. The Hilbert Series of the second quiver reads

$$\begin{aligned} \text{HS}[T_{(3,2)}[(SU(5))]](p_1, p_2, n_1, n_2, n_3, n_4, 0) = \\ t^{4n_1+2n_2-2n_4} (1-t^2)^5 K_{(3,2)}^{U(5)}(p_1, p_2, t) \Psi_{U(5)}^{(n_1, n_2, n_3, n_4, 0)}(p_1 t^2, p_1, p_1 t^{-2}, p_2 t, p_2 t^{-1}; t), \end{aligned}$$

where the fugacities satisfy the constraint $p_1^3 p_2^2 = 1$ and we set $n_5 = 0$. We glue together the three quiver diagrams gauging the common $SU(5)$ global symmetry group using the formula (15). The corresponding unrefined Hilbert Series reads

$$\text{HS}_{SU(10)}(t, 1, \dots, 1) = \frac{P_{SU(10)}(t)}{(1-t)^{52}(1+t)^{32}(1+t+t^2)^{26}}, \quad (\text{B.15})$$

where $P_{SU(10)}(t)$ is a palindromic polynomial given by

$$\begin{aligned} P_{SU(10)}(t) = & 1 + 6t + 88t^2 + 684t^3 + 5068t^4 + 33270t^5 + 195032t^6 + 1042038t^7 + 5115964t^8 + \\ & 23174044t^9 + 97503058t^{10} + 382784562t^{11} + 1407298803t^{12} + 4861817820t^{13} + 15830225132t^{14} + \\ & 48702053512t^{15} + 141895604363t^{16} + 392320199370t^{17} + 1031217756368t^{18} + 2581111810032t^{19} + \\ & 6161027223819t^{20} + 14043326715580t^{21} + 30604355641425t^{22} + 63836857086540t^{23} + \\ & 127576383100320t^{24} + 244498094132778t^{25} + 449725582642239t^{26} + 794538443153332t^{27} + \\ & 134919812455657t^{28} + 2203437309257322t^{29} + 3462860983386664t^{30} + 5239624609952376t^{31} + \\ & 7636510804862128t^{32} + 10725001017564682t^{33} + 14519955240446539t^{34} + 18955608867282408t^{35} + \\ & 23869071305501125t^{36} + 28997597428531974t^{37} + 33994115765048473t^{38} + 38461909792331794t^{39} + \\ & 42004659363999571t^{40} + 44283517877832144t^{41} + 45070023311322202t^{42} + 44283517877832144t^{43} + \\ & \dots \text{palindrome} \dots + t^{84}. \end{aligned} \quad (\text{B.16})$$

The dimension of the pole of the unrefined HWG (72) at $t = 1$ is 7 while the degree of the polynomial arising from the dimension of the $SU(10)$ representation $[n_1, n_3, n_5, n_6, n_2 + n_4, n_6, n_5, n_3, n_1]$ is 45. They add up to 52 which is the dimension of the pole at $t = 1$ of the unrefined Hilbert Series (B.16).

B.5 $N = 5$, $G_{\text{global}} = SU(2) \times SU(10)$

The ‘‘building blocks’’ of the quiver diagram reported in figure 12 are the quiver diagram [5]-(4)-(3)-(2)-(1) and the quiver diagram [5]-(2)-(1). We use the formula (7) and we compute the Hilbert Series of the first quiver diagram

$$\begin{aligned} \text{HS}[\text{T}_{(1,1,1,1,1)}[(SU(5))](x_1, x_2, x_3, x_4, x_5, n_1, n_2, n_3, n_4, 0) = \\ = t^{4n_1+2n_2-2n_4}(1-t^2)^5 \text{PE} \left[\sum_{i=1}^5 \sum_{j=1}^5 \frac{x_i}{x_j} t^2 \right] \Psi_{U(5)}^{(n_1, n_2, n_3, n_4, 0)}(x_1, x_2, x_3, x_4, x_5; t), \end{aligned} \quad (\text{B.17})$$

where the fugacities x_i satisfy the constraint $x_1 x_2 x_3 x_4 x_5 = 1$ and we set $n_5 = 0$. The Hilbert Series of the second quiver reads

$$\begin{aligned} \text{HS}[\text{T}_{(3,1,1)}(SU(5))](p_1, p_2, p_3, n_1, n_2, n_3, n_4, 0) = \\ = t^{4n_1+2n_2-2n_4}(1-t^2)^5 K_{(3,1,1)}^{U(5)}(p_1, p_2, p_3, t) \Psi_{U(5)}^{(n_1, n_2, n_3, n_4, 0)}(p_1 t^2, p_1, p_1 t^{-2}, p_2, p_3; t), \end{aligned}$$

where this time the fugacities satisfy the constraint $p_1^3 p_2 p_3 = 1$ and we set $n_5 = 0$. We use the formula (15) and we compute the Hilbert Series of the quiver reported in figure 12. The corresponding unrefined Hilbert series reads

$$\text{HS}_{SU(2) \times SU(10)}(t, 1, \dots, 1) = \frac{P_{SU(2) \times SU(10)}(t)}{(1-t^2)^{54}(1+t^2)^{27}}, \quad (\text{B.18})$$

where $P_{SU(2) \times SU(10)}(t)$ is a palindromic polynomial given by

$$\begin{aligned}
P_{SU(2) \times SU(10)}(t) = & 1 + 75t^2 + 3227t^4 + 93628t^6 + 1995005t^8 + 32743316t^{10} + 428018495t^{12} + \\
& 4567431397t^{14} + 40562120142t^{16} + 304454872921t^{18} + 1956103026075t^{20} + 10872201648590t^{22} + \\
& 52743372310579t^{24} + 225023527079799t^{26} + 849778417559022t^{28} + 2856352728077173t^{30} + \\
& 8586607150389539t^{32} + 23180525972202894t^{34} + 56397132911896665t^{36} + 124035385935159636t^{38} + \\
& 247240406636319041t^{40} + 447648087747618549t^{42} + 737563815297879763t^{44} + 1107559743906553834t^{46} + \\
& 1517629622095781174t^{48} + 1899315502205824974t^{50} + 2172420593881086252t^{52} + 2271821768928281868t^{54} + \\
& \dots \text{palindrome} + \dots + t^{108}.
\end{aligned} \tag{B.19}$$

We observe that the dimension of the pole of the unrefined HWG (83) at $t = 1$ is 8 while the degree of the polynomial arising from the dimension of the $SU(2) \times SU(10)$ representation $[2n_2 + n_4; n_1, n_3, n_5, n_6, n_4 + 2n_7, n_6, n_5, n_3, n_1]$ is 46. They add up to 54 which is the dimension of the pole at $t = 1$ of the unrefined Hilbert Series.

B.6 $N = 6$, $G_{global} = SU(12)$

The ‘‘building blocks’’ of the quiver diagram reported in fig.13 are the quiver diagram [6]-(5)-(4)-(3)-(2)-(1) and the quiver diagram [6]-(2). We use the formula (7) and we compute the corresponding Hilbert Series. The HS of the first quiver reads

$$\begin{aligned}
& \text{HS}[\mathbb{T}_{(1,1,1,1,1,1)}(SU(6))](x_1, x_2, x_3, x_4, x_5, x_6, n_1, n_2, n_3, n_4, n_5, 0) = \\
& = t^{5n_1+3n_2+n_3-n_4-3n_5-5n_6} (1-t^2)^6 \text{PE} \left[\sum_{i=1}^6 \sum_{j=1}^6 \frac{x_i}{x_j} t^2 \right] \Psi_{U(6)}^{(n_1, n_2, n_3, n_4, n_5, 0)}(x_1, x_2, x_3, x_4, x_5, x_6; t),
\end{aligned} \tag{B.20}$$

where the fugacities x_i satisfy the constraint $x_1 x_2 x_3 x_4 x_5 x_6 = 1$ and we set $n_6 = 0$. The Hilbert Series of the second quiver reads

$$\begin{aligned}
& \text{HS}[\mathbb{T}_{(4,2)}(SU(6))](p_1, p_2, p_3, n_1, n_2, n_3, n_4, n_5, 0) = t^{5n_1+3n_2+n_3-n_4-3n_5-5n_6} (1-t^2)^6 \\
& K_{(4,2)}^{U(6)}(p_1, p_2, t) \Psi_{U(6)}^{(n_1, n_2, n_3, n_4, n_5, 0)}(p_1 t^3, p_1 t^1, p_1 t^{-1}, p_1 t^{-3}, p_2 t^1, p_2 t^{-1}; t),
\end{aligned}$$

where the fugacities satisfy the constraint $p_1^4 p_2^2 = 1$ and we set $n_6 = 0$.

B.7 $N = 6$, $G_{global} = SU(2) \times SU(12)$

The ‘‘building blocks’’ of the quiver diagram reported in fig.14 are the quiver diagram [6]-(5)-(4)-(3)-(2)-(1) and the quiver diagram [6]-(2). We use the formula (7) and we find the Hilbert Series for each of them. The Hilbert Series of the first quiver diagram reads

$$\begin{aligned}
& \text{HS}[\mathbb{T}_{(1,1,1,1,1,1)}(SU(6))](x_1, x_2, x_3, x_4, x_5, x_6, n_1, n_2, n_3, n_4, n_5, 0) = \\
& = t^{5n_1+3n_2+n_3-n_4-3n_5-5n_6} (1-t^2)^6 \text{PE} \left[\sum_{i=1}^6 \sum_{j=1}^6 \frac{x_i}{x_j} t^2 \right] \Psi_{U(6)}^{(n_1, n_2, n_3, n_4, n_5, 0)}(x_1, x_2, x_3, x_4, x_5, x_6; t),
\end{aligned} \tag{B.21}$$

where the fugacities x_i satisfy the constraint $x_1 x_2 x_3 x_4 x_5 x_6 = 1$ and we set $n_6 = 0$. The HS of the second quiver diagram is

$$\begin{aligned}
& \text{HS}[\mathbb{T}_{(4,1,1)}(SU(6))](p_1, p_2, p_3, n_1, n_2, n_3, n_4, n_5, 0) = t^{5n_1+3n_2+n_3-n_4-3n_5-5n_6} (1-t^2)^6 \\
& K_{(4,1,1)}^{U(6)}(p_1, p_2, p_3, t) \Psi_{U(6)}^{(n_1, n_2, n_3, n_4, n_5, 0)}(p_1 t^3, p_1 t, p_1 t^{-1}, p_1 t^{-3}, p_2, p_3; t),
\end{aligned}$$

where the fugacities satisfy the constraint $p_1^4 p_2 p_3 = 1$ and we set $n_6 = 0$. We glue together the Hilbert Series of the three quivers diagrams using the formula (15).

B.8 $G_{global} = SO(14) \times U(1)$

The quiver diagram with $SO(14) \times U(1)$ global symmetry is reported in figure 16.

The “building blocks” of this theory are the quiver diagram [5]-(4)-(3)-(2)-(1) and the quiver diagram [5]-(3)-(1). We use the formula (7) and we find the HS of each “building blocks”. The HS of the quiver [5]-(4)-(3)-(2)-(1) reads

$$\begin{aligned} & \text{HS}[\text{T}_{(1,1,1,1,1)}(SU(5))](x_1, x_2, x_3, x_4, x_5, n_1, n_2, n_3, n_4, 0) = \\ & = t^{4n_1+2n_2-2n_4} (1-t^2)^5 \text{PE} \left[\sum_{i=1}^5 \sum_{j=1}^5 \frac{x_i}{x_j} t^2 \right] \Psi_{U(5)}^{(n_1, n_2, n_3, n_4, 0)}(x_1, x_2, x_3, x_4, x_5; t), \end{aligned} \quad (\text{B.22})$$

where the fugacities x_i satisfy the constraint $x_1 x_2 x_3 x_4 x_5 = 1$ and we set $n_5 = 0$. The HS of the quiver [5]-(3)-(1) reads

$$\begin{aligned} & \text{HS}[\text{T}_{(2,2,1)}(SU(5))](y_1, y_2, y_3, n_1, n_2, n_3, n_4, 0) = \\ & = t^{4n_1+2n_2-2n_4} (1-t^2)^5 K_{(2,2,1)}^{U(5)}(y_1, y_2, y_3, t) \Psi_{U(5)}^{(n_1, n_2, n_3, n_4, 0)}(y_1, y_2, y_3; t), \end{aligned} \quad (\text{B.23})$$

where the fugacities y_i satisfy the constraint $y_1^2 y_2^2 y_3 = 1$ and we set $n_5 = 0$. The factor $K_{(2,2,1)}^{U(5)}$ is given by

$$K_{(2,2,1)}^{U(5)}(y_1, y_2, y_3, t) = \text{PE} \left[3t^2 + 2t^4 + \frac{t^2 y_1}{y_2} + \frac{t^4 y_1}{y_2} + \frac{t^2 y_2}{y_1} + \frac{t^4 y_2}{y_1} + \frac{t^3 y_1}{y_3} + \frac{t^3 y_2}{y_3} + \frac{t^3 y_3}{y_1} + \frac{t^3 y_3}{y_2} \right].$$

We glue together the three gauge theories using the formula (15). The corresponding unrefined HS reads

$$\begin{aligned} & \text{HS}_{SO(14) \times U(1)}(t) = \\ & = \frac{1}{(1-t)^{44} (1+t)^{36} (+1+t+t^2)^{22}} (1 + 14t + 161t^2 + 1450t^3 + 11235t^4 + 76076t^5 + 460270t^6 + \\ & 2515464t^7 + 12543757t^8 + 57485438t^9 + 243590526t^{10} + 959135368t^{11} + 3524162306t^{12} + \\ & 12127021450t^{13} + 39204806082t^{14} + 119399888418t^{15} + 343401526770t^{16} + 934694624360t^{17} + \\ & 2412360246363t^{18} + 5913823733836t^{19} + 13791838727022t^{20} + 30641576394730t^{21} + \\ & 64935808673035t^{22} + 131413027444364t^{23} + 254228078480137t^{24} + 470594664975578t^{25} + \\ & 834214522978169t^{26} + 1417259205143370t^{27} + 2309205745322137t^{28} + 3610677213615316t^{29} + \\ & 5420885724624754t^{30} + 7818510084828412t^{31} + 10837817133061051t^{32} + 14444157990254486t^{33} + \\ & 18514905671456643t^{34} + 22832559942436902t^{35} + 27095427129072378t^{36} + 30947837022133142t^{37} + \\ & 34026931702773205t^{38} + 36018018650672498t^{39} + 36707016757599132t^{40} + \dots + \text{palindrome} + t^{80}). \end{aligned} \quad (\text{B.24})$$

The dimension of the pole arising from the HWG (129) is 5 while the degree of the polynomial arising from the $[0, n_2, 0, n_4, 0, n_6, n_7]$ representation is 39. They add up to 44 which is the dimension of the pole of the unrefined HS (B.24).

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